Quantum Field Theory
Exercises 6

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Gauge Fields

1. Demonstrate that, for a matrix U, defined as

$$U = e^{M}\,, \tag{1}$$

one has

$$\det U = e^{\mathrm{Tr}M}\,. \tag{2}$$

2. The $SU(N)$ Lie algebra is defined by

$$[\boldsymbol{T}^a, \boldsymbol{T}^b] = i f^{abc} \boldsymbol{T}^c\,, \tag{3}$$

where f_{abc} is the antisymmetric structure constant tensor

$$f_{abc} = -f_{bac}\,, \tag{4}$$

which is a real object, and the generators are Hermitian

$$\boldsymbol{T}^{a\,\dagger} = \boldsymbol{T}^a\,. \tag{5}$$

Show that

$$\left(\boldsymbol{T}^a \boldsymbol{T}^b\right)^* = \boldsymbol{T}^b \boldsymbol{T}^a\,. \tag{6}$$

Knowing that $\mathrm{Tr}(\boldsymbol{T}^a\boldsymbol{T}^b) = T_F \delta_{ab}$, with $T_F = \frac{1}{2}$, derive

$$\mathrm{Tr}(\boldsymbol{T}^a\boldsymbol{T}^b\boldsymbol{T}^b\boldsymbol{T}^a) = C_A C_F^2\,, \tag{7}$$

$$\mathrm{Tr}(\boldsymbol{T}^a\boldsymbol{T}^b\boldsymbol{T}^a\boldsymbol{T}^b) = -\frac{1}{2} C_F\,, \tag{8}$$

where we defined the Casimir operators

$$C_F = \frac{N^2-1}{2N}\,, \qquad C_A = N\,. \tag{9}$$

3. By construction, under the gauge transformation

$$U(x) = e^{i\alpha(x)}\,, \tag{10}$$

where $\alpha(x) = \theta(x)$ for the abelian and $\alpha(x) = \theta_a(x) T^a$ for the non-abelian case, a covariant derivative of a Dirac field transforms as the field itself

$$\Psi(x) \to U(x)\Psi(x)\,, \tag{11}$$

$$D_\mu \Psi(x) \to U(x) D_\mu \Psi(x)\,. \tag{12}$$

Using this property, calculate the commutators of covariant derivatives

$$[D_\mu, D_\nu]\,, \tag{13}$$

for QED and QCD and relate them with rank-two tensors introduced for those theories during the lecture.