Quantum Field Theory
Exercises 5

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1 Dirac Field

1. Starting from the generating functional

$$Z[\eta, \bar{\eta}] = \int D\Psi D\bar{\Psi} \, \exp\left\{ i \int d^4x \left[\bar{\Psi}(i\not{\partial} - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta \right] \right\} , \tag{1.1}$$

where $\eta, \bar{\eta}, \Psi, \bar{\Psi}$ are Grassmann variables, transform it to the form

$$Z[\eta, \bar{\eta}] = C \exp\left\{ -i \int d^4x d^4y \, \bar{\eta}(x) \hat{K}^{-1}(x, y) \eta(y) \right\} . \tag{1.2}$$

Determine C and $\hat{K}$.

2. Dirac spinors can be represented as

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \, \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \, \xi^s \end{pmatrix} , \tag{1.3}$$

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \, \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \, \eta^s \end{pmatrix} , \tag{1.4}$$

where ξ and η are 2-component vectors. It is convenient to choose the basis vectors such that

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad \eta^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad \eta^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{1.5}$$

Then, ξ^1 and ξ^2 correspond to spin up and spin down particle, respectively, while η^1 and η^2 describe spin up and spin down antiparticle.

Using the above, derive the following spin sums:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \, , \tag{1.6}$$

$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \, . \tag{1.7}$$

3. Calculate

$$g^{\mu\nu} g_{\nu\sigma} \,, \tag{1.8}$$

$$g^{\mu\nu} g_{\nu\mu} \,, \tag{1.9}$$

$$\mathrm{Tr}[\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}] \,, \tag{1.10}$$

$$\mathrm{Tr}[\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\rho}] \,. \tag{1.11}$$