MONTE CARLO METHODS

FOR HIGH ENERGY PHYSICS

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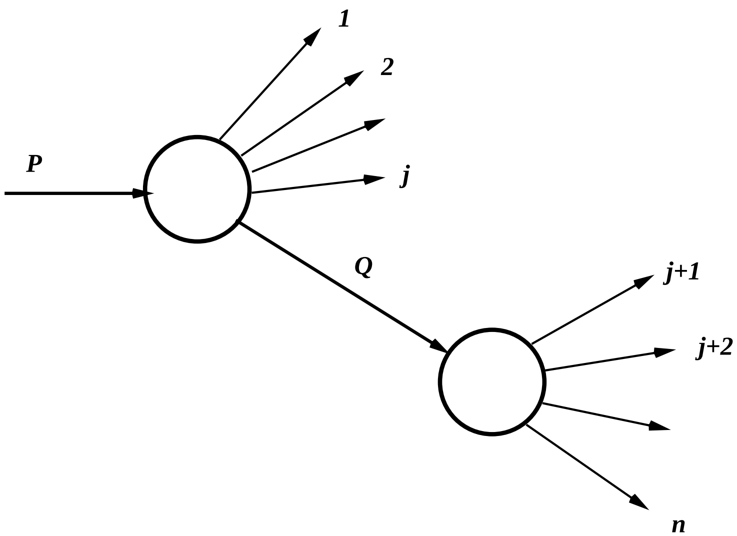
Lecture 4

Slides, program sources: http://home.cern.ch/jadach

Outline:

1. Cascade parametrization of the phase space (for multibranching MC's).
2. Lorentz group parametrization of the phase space.

Most common parametrization of the phase space in MC's



Phase space parametrization in most of the MC's is based on the "cascade identity".

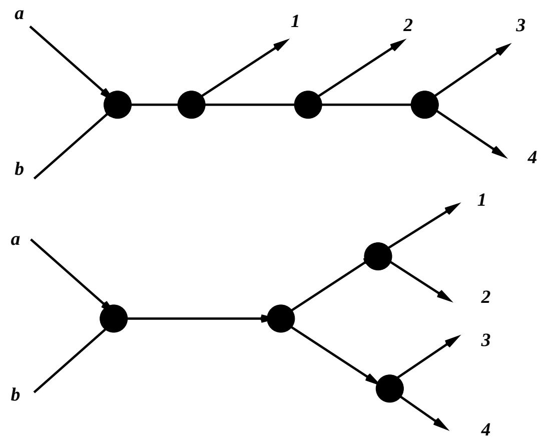
Defining LORENTZ INVARIANT PHASE SPACE as:

$$d\tau_n(P;p_1,p_2,...,p_n) \equiv \delta^{(4)}(P-\sum_{j=1}^{n} p_j)\prod_{j=1}^{n}\frac{d^3p_j}{2p_j^0}$$

the identity is:

$$\int d\tau_n(P;p_1,p_2,...,p_n) =$$
$$= \int ds_Q \int d\tau_j(P;p_1,p_2,...,p_j,Q) \int d\tau_{n-j}(Q;p_{j+1},j+2,...,p_n)$$

Most common cascade parametrization of the phase space in MC's



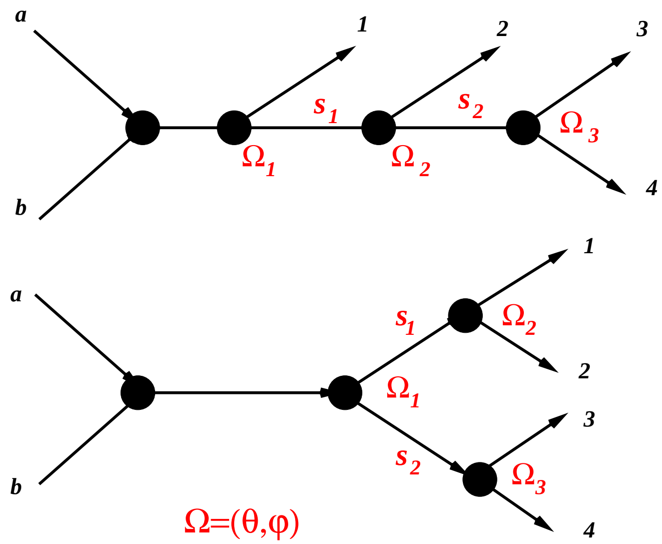
Employing cascade identity several times we may always decompose n-body phase space into cascade of the 2-body phase spaces.

For example in case of 4 final particles there are 2 nontrivial cascades (kinematic trees) and about 50 permutations altogether.

(This is what is in KORALW MC or EXCALIBUR type programs.)

Note that even for multiperipheral diagram the trivial tree is adequate (we can find relation between angle and the transfer, Kajantie Byckling 1965).

Cascade parametrization of the phase space in MC's



In this case phase space is parametrized in terms of 2 masses and 6 angles. For each tree (including permutations) one may knows from Feynman diagram possible menu of singularities for effective masses $1, 1/s, 1/((s-M^2)^2+M^2\Gamma^2)...$ and for θ angle $1, 1/t, 1/t^2, 1/(t-M^2)...$

This leads to many branches in the MC. Each of them represents one kinemaical tree and one assignment of singularities.

Thanks to "slight simplifications" countered by the correcting weights we can generate $(s_1, s_2, \Omega_1, \Omega_2, \Omega_3)$ by analytical mapping.

This is how it looks in many MC programs. But not in all of them, see next slides...

Lorentz group parametrization of the phase space

There is an interesting and powerful extension of the cascade parametrization, which relies strictly on the "rigid body" parametrization on the group of the four momenta.

It allows to "attach the Lorentz frame" to a group of momenta.

These momenta can be timelike (s-channel) or spacelike (t-channel).

This technique is particularly useful if we want to have 100% control of spin poplarization effects for the multibody final state, because it precisely keeps track of all "azimuthal angles" which influence the phases of the helicity amplitudes.

In the following I shall "give the taste" of what it is.

Only s-channel version is shown here

(t-channel version is a bit more complicated.)

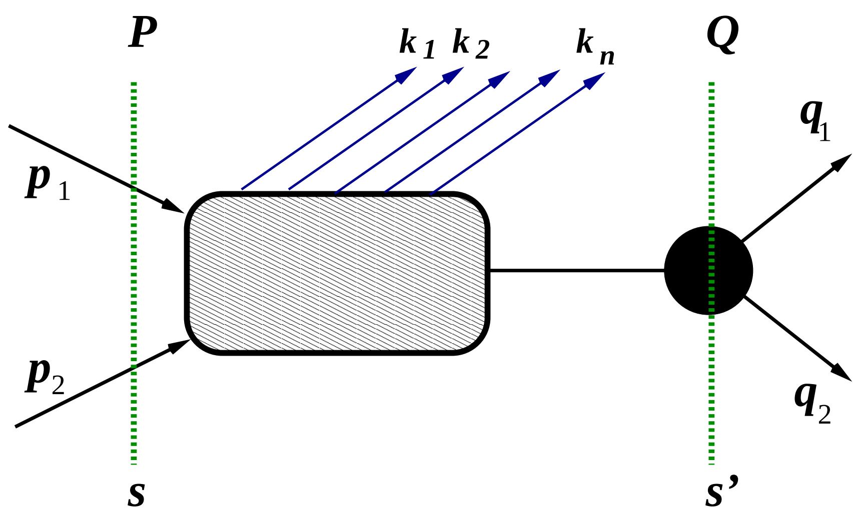
This s-channel identity was employed for the first time in the YFS3 MC event generator in 1988/89. (t-channel version in BHLUMI).

Lorentz group parametrization of the phase space

$$\sigma_n = \int \tau_{n+2}(p_1+p_2;\ k_1,\dots,k_n,\ q_1,q_2)\ \Sigma_n(p_i,q_j,k_l)$$

Lets introduce several "spurious" integrations countered by delta functions:

$$\sigma_n = \int d^4P \int ds' \int \prod_{j=1}^{n} \frac{d^3k_j}{2k_j^0} \int d^4Q \int \frac{d^3q_1}{2q_1^0} \frac{d^3q_2}{2q_2^0}\ \delta^{(3)}(P)\ \delta^+(s-P^2)$$
$$\delta^{(4)}\left(P-Q-\sum_{j=1}^{n} k_j\right)\ \delta^{(4)}(Q-q_1-q_2)\ \delta^+(Q^2-s')\ 2s^{1/2}\Sigma_n(p_i,q_j,k_l)$$



Lets go!

Even more integrations and delta's are added implanting the identity:

$$\int d\cos\omega d\psi\; 2\delta^{+}\left(\frac{X^2}{s_Q}-1\right)\frac{d^4X}{s_Q^2}\;\delta^{(3)}\left(L_A^{-1}\frac{Q}{\sqrt{s_Q}}\right)\;\delta^{\{12\}}\left(L_A^{-1}\frac{q_1}{|\vec{q}_1|_Q}\right)\;\equiv 1.$$

where the big transformation from the rests frame of $Q = q_1 + q_2$ to LAB enters:

$$L_A = R(\psi,\omega)B_X = B_{RX}R,\quad R = R(\psi,\omega) = R_3(\psi)R_2(\omega)$$

$$L_A^{-1} = B_X^{-1}R^{-1}(\psi,\omega) = R^{-1}B_{RX}^{-1}$$

After that the integral looks as follows (all variables are integration variables!):

$$\sigma_n = \int ds' \int d\cos\omega d\psi\; 2\delta^{+}\left(\frac{X^2}{s'}-1\right)\frac{d^4X}{s'^2}$$

$$\int d^4P\;\int\;\prod_{j=1}^{n}\frac{d^3k_j}{2k_j^0}\;\int d^4Q\;\int\frac{d^3q_1}{2q_1^0}\;\frac{d^3q_2}{2q_2^0}\;\delta^{(3)}(P)\;\delta^{+}(s-P^2)$$

$$\delta^{(4)}\left(P-Q-\sum_{j=1}^{n}k_j\right)\;\delta^{(4)}(Q-q_1-q_2)\;\delta^{+}(Q^2-s')$$

$$\delta^{(3)}\left(L_A^{-1}\frac{Q}{\sqrt{s_Q}}\right)\;\delta^{\{12\}}\left(L_A^{-1}\frac{q_1}{|\vec{q}_1|_Q}\right)\;2s^{1/2}\Sigma_n(p_i,q_j,k_l)$$

Here $R_i(x)$ are matrices of rotations around $i-th$ axis and B_X is matrix of the "parallel boost" transformation defined by timelike vector X, see next slide.

The Parallel Lorentz boost

from to the rest frame of the timelike 4-momentum $q = (q^0, \vec{q})$, $q^2 = m^2$ to LAB:

$$B_q \equiv \begin{bmatrix} \frac{q^0}{m}, & \frac{\vec{q}^T}{m} \\ \frac{\vec{q}}{m}, & I + \frac{\vec{q}\otimes\vec{q}}{m(m+q^0)} \end{bmatrix} \equiv \begin{bmatrix} \cosh\eta, & \vec{n}^T \sinh\eta \\ \vec{n}\sinh\eta, & I + \vec{n}\otimes\vec{n}\,(\cosh\eta - 1) \end{bmatrix},$$

where T marks matrix and $\otimes$ tensor product, $\vec{n} = \vec{q}/|\vec{q}|$, $\cosh\eta = q^0/m$.

Basic properties and relations:

$B_q \overset{\circ}{q} = q, \quad \overset{\circ}{q} \equiv (m, 0, 0, 0),$

$B_{B_3(\eta)\overset{\circ}{q}} = B_3(\eta),$

$B_q^{-1} = B_{q^P}, \quad q^P = (q^0, -\vec{q}),$

$RB_qR^{-1} = RB_qR^T = B_{Rq},$

$RB_q = B_{Rq}R,$ where R is any 3-dimensional rotation.

For $q = R\hat{q}_z = R_3(\phi)R_2(\theta)q_z;\quad q_z = (q^0, 00, 0, |\vec{q}|) = B_3(\eta)\overset{\circ}{q}$, we have:

$B_q = B_{Rq_z} = RB_{q_z}R^{-1} = R_3(\phi)R_2(\theta)B_3(\eta)R_2(-\theta)R_3(-\phi)$

The Parallel Lorentz boost

The following identity holds:

$$\int 2\delta^{+}\left(\frac{X^2}{m^2}-1\right)\ \frac{d^4X}{m^4}\ \delta^{(3)}\left(L_X^{-1}\frac{p}{m}\right)\ f(X,q) = f(p,q)$$

Proof exploit (hyper) spherical parametrization of p:

$$\int \frac{2}{m^2}\delta^{+}(X^2-m^2)d^4X\ \delta^{(3)}(L_X^{-1}\frac{p}{m})\ f(X,q)$$
$$= \int \sinh^2\eta_X\, d\eta_X\, d\cos\theta_X\, d\phi_X\ \delta^{(3)}(L_X^{-1}\frac{p}{m})\ f(X,q)$$
$$= \int \sinh^2\eta_X\, d\eta_X\, d\cos\theta_X\, d\phi_X\ \delta^{(3)}(B_3(-\eta_X)R_3(\phi_X)R_2(-\theta_X)\frac{p}{m})\ f(X,q)$$
$$= f(\hat{X}(p),q) = f(p,q).$$

One can also keep boost vector $X = x$ normalized to one:

$$\int 2\delta^{+}\left(x^2-1\right)\ d^4x\ \delta^{(3)}\left(L_x^{-1}\frac{p}{m}\right)\ f(mx,q) = f(p,q)$$

Change of variables

Now we change of variables in the integral which is just identical to the Lorentz transformation L_A; it formally looks as follows:

$$\boxed{P \to L_A P,\ Q \to L_A Q,\ k_j \to L_A k_j,\ q_i \to L_A q_i,}$$

First we do a substitution $p = L_A p'$; the new variables are $p' = L_A^{-1} p$. Next we rename $p' \to p$. All parameters in L_A are passive variables of the outer integrations.

In most places L_A drops out because of Lorentz covariance:

$$\sigma_n = \int ds' \int d\cos\omega d\psi\ 2\delta^+\left(\frac{X^2}{s'} - 1\right) \frac{d^4X}{s'^2}$$
$$\int d^4P \int \prod_{j=1}^n \frac{d^3k_j}{2k_j^0} \int d^4Q \int \frac{d^3q_1}{2q_1^0} \frac{d^3q_2}{2q_2^0}\ \delta^{(3)}(L_A P)\ \delta^+(s - P^2)$$
$$\delta^{(4)}\left(P - Q - \sum_{j=1}^n k_j\right)\ \delta^{(4)}(Q - q_1 - q_2)\ \delta^+(Q^2 - s')$$
$$\delta^{(3)}\left(\frac{q_1+q_2}{\sqrt{s'}}\right)\ \delta^{\{12\}}\left(\frac{q_1}{|\vec{q}_1|_Q}\right)\ 2s^{1/2}\Sigma_n(p_i, q_j, k_l)$$

Now we reduce most of δ-functions staring with the easiest...

Momenta q_1 and q_2 are fixed!

First we apply identity:

$$\int d^4Q\, \delta^+(Q^2 - s')\, \int \frac{d^3q_1}{2q_1^0} \frac{d^3q_2}{2q_2^0}\, \delta^{(4)}(Q - q_1 - q_2)$$
$$\delta^{(3)}\left(\frac{q_1+q_2}{\sqrt{s'}}\right)\, \delta^{\{12\}}\left(\frac{q_1}{|\vec{q}_1|_Q}\right) = s'\,\frac{1}{2}\beta(s'; q_1^2, q_2^2),$$

to our integral:

$$\sigma_n = \int ds' \int d\cos\omega d\psi\; 2\delta^+\left(\frac{X^2}{s'} - 1\right) \frac{d^4X}{s'^2}$$
$$\int d^4P \int \prod_{j=1}^n \frac{d^3k_j}{2k_j^0} \int d^4Q \int \frac{d^3q_1}{2q_1^0} \frac{d^3q_2}{2q_2^0}\, \delta^{(3)}(L_A P)\, \delta^+(s - P^2)$$
$$\delta^{(4)}\left(P - Q - \sum_{j=1}^n k_j\right)\, \delta^{(4)}(Q - q_1 - q_2)\, \delta^+(Q^2 - s')$$
$$\delta^{(3)}\left(\frac{q_1+q_2}{\sqrt{s'}}\right)\, \delta^{\{12\}}\left(\frac{q_1}{|\vec{q}_1|_Q}\right)\, 2s^{1/2}\Sigma_n(p_i, q_j, k_l)$$

where

$$q_1 = \hat{q}_1 = \left(0, 0, |\vec{q}_1|_Q, \sqrt{s'}/2\right), \quad q_1 = \hat{q}_1 = \left(0, 0, -|\vec{q}_1|_Q, \sqrt{s'}/2\right)$$
$$\vec{q}_1|_Q = \frac{\sqrt{s'}}{2}\, \beta(s'; q_1^2, q_2^2) = \frac{\sqrt{s'}}{2}\, \beta_f.$$

are the solutions of constraints imposed by the reduced δ's!!!.

The net result of the first reduction is:

$$\sigma_n = \int d^4P \int ds' \int d\cos\omega d\psi\; 2\delta^+\left(\frac{X^2}{s'}-1\right)\frac{d^4X}{s'^2}\int \prod_{j=1}^n \frac{d^3k_j}{2k_j^0}$$
$$\delta^{(3)}(L_A P)\; \delta^+(s-P^2)\; \delta^{(4)}\left(P-\hat{Q}-\sum_{j=1}^n k_j\right)\; 2s^{1/2}\frac{s'\beta_f}{16}\Sigma_n(p_i,q_j,k_l)$$ **is**

exactly what we wanted, because at this point $\hat{q}_i$ are not the integration variables, they are fixed as the constant vectors depending only on s'!

We denote: $\hat{Q} \equiv \hat{q}_1 + \hat{q}_1 = (0,0,0,\sqrt{s'})$.

The next step is rather trivial – we simply apply:

$$\int d^4P\; \delta^+(s-P^2)\; \delta^{(4)}\left(P-\hat{Q}-\sum_{j=1}^n k_j\right) = \delta^+(s-(\hat{Q}+\sum_{j=1}^n k_j)^2)$$

arriving to

$$\sigma_n = \int ds' \int d\cos\omega d\psi \int \prod_{j=1}^n \frac{d^3k_j}{2k_j^0}\;\; 2\delta^+\left(\frac{X^2}{s'}-1\right)\frac{d^4X}{s'^2}$$
$$\delta^{(3)}\left(L_A(\hat{Q}+\sum_{j=1}^n k_j)\right)\; \delta^+\left(s-(\hat{Q}+\sum_{j=1}^n k_j)^2\right)\; \frac{s'\beta_f}{8}s^{1/2}\Sigma_n(p_i,q_j,k_l).$$

Here comes more interesting reduction:

$$\int 2\delta^+(\frac{X^2}{s'}-1)\frac{d^4X}{s'^2}\delta^{(3)}(L_A\hat{P}) = \int 2\delta^+(\frac{X^2}{s'}-1)\frac{d^4X}{s'^2}\delta^{(3)}((B_{XP})^{-1}\hat{P}) \equiv s^{-3/2},$$

and $\hat{P} = \hat{Q}+\sum_{j=1}^n k_j$ is initial 4-mom., as seen from rest frame of Q.

In the process of reducing δ's parameters in L_A got completely defined:

$$k_i|_{CMS} = L_A k_i,\quad q_i|_{CMS} = L_A\hat{q}_i,$$
$$L_A = R_3(\psi)R_2(\omega)B_{\hat{P}P} = R_3(\psi)R_2(\omega)B^{-1}_{\hat{P}}.$$

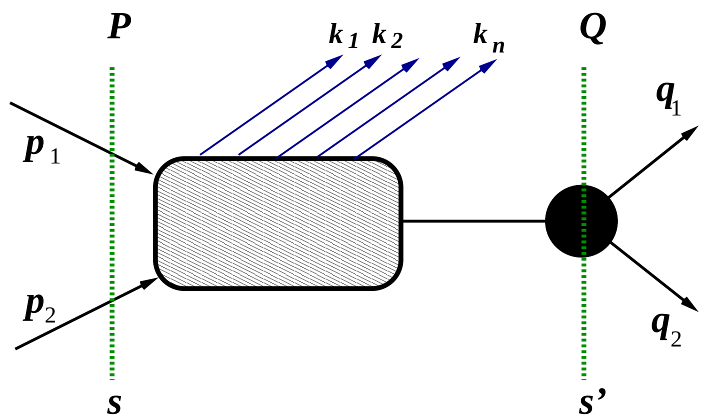
Lorentz group parametrization

Final results is very simple. Jacobian is, however, reminding us of the fact that we did something non-trivial:

$$\sigma_n = \int ds' \int d\cos\omega d\psi \int \prod_{j=1}^{n} \frac{d^3 k_j}{2k_j^0} \delta\big(s - (\hat{Q} + \sum_{j=1}^{n} k_j)^2\big) \frac{s'}{2s}\beta(s', q_1^2, q_2^2) \ \Sigma_n(p_i, q_j, k_l).$$

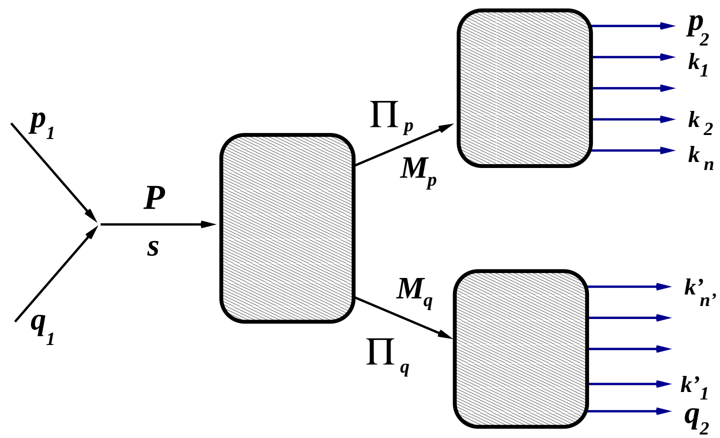
and

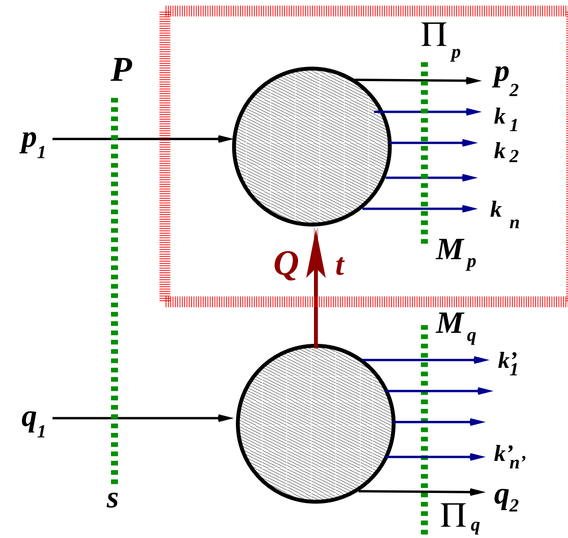
$$k_i|_{CMS} = L_A k_i, \quad q_i|_{CMS} = L_A \hat{q}_i,$$
$$L_A = R_3(\psi)R_2(\omega)B_{\hat{P}P} = R_3(\psi)R_2(\omega)B_{\hat{P}}^{-1},$$
$$\hat{P} = \hat{Q} + \sum_{j=1}^{n} k_j, \qquad \hat{Q} \equiv \hat{q}_1 + \hat{q}_1 = (0, 0, 0, \sqrt{s'})$$



Note: Angles ω, ψ should not be interpreted as angles of some patricle in some frame, they are Euler-type "rigid-body" parametrization the system (q_1, q_2).

t-channel case





$$\sigma_{n,n'} =$$

$$\int d\tau_{n+n'+2}(p_1+q_1;\; p_2, q_2,\; k_1,\dots,k_n,\; k'_1,\dots,k_{n'})\Sigma_{n,n'}(p_i,q_j,k_l,k'_j)$$

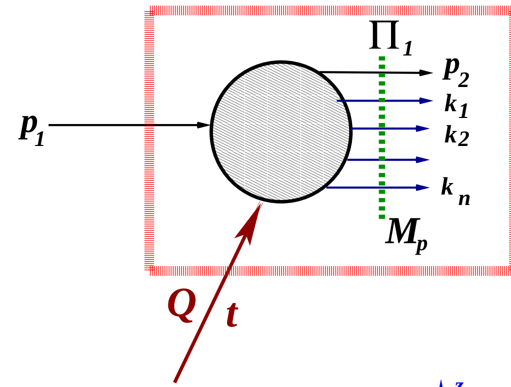
$$= \int dM_p^2\; dM_q^2\; \int d\tau_2(p_1+q_1;\Pi_p,\Pi_q)$$

$$\int d\tau_{n+1}(\Pi_p;k_1,\dots,k_n)\; \int d\tau_{n'+1}(\Pi_q;k'_1,\dots,k_{n'})\; \Sigma_{n,n'}(p_i,q_j,k_l,k'_j)$$

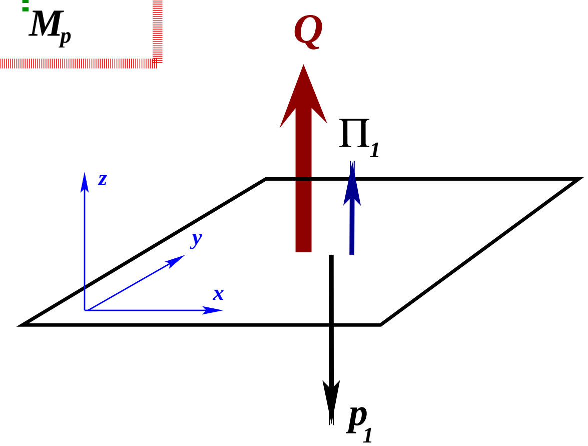
The two body kinematics is fully “resolvable” in term of transfer $t = Q^2$ and of the azimuthal angle φ: $\int d\tau_2(p_1+q_1;\Pi_p,\Pi_q) = \int_0^{2\pi} d\varphi \frac{dt}{s}$, provided masses $\Pi_p^2 = M_p^2$ $\Pi_q^2 = M_q^2$ are known.

We concentrate from now on, on part of the phase space inside the “red box”.

Breit frame (rest of Q)



$$Q + p_q = \Pi_p,$$
$$\hat{Q} = (0, 0, 0, \sqrt{|t|}),$$
$$\hat{p}_1 = (\hat{p}_1^0, 0, 0, \hat{p}_1^3),$$
$$\hat{\Pi}_p^0 = \hat{p}_1^0 = \frac{\sqrt{|t|}}{2}\beta(t, M_p^2, m_1^2),$$
$$\hat{p}_1^3 = \frac{t+m_1^2-M_p^2}{2\sqrt{|t|}},$$
$$\hat{\Pi}_p^3 = \frac{-t+m_1^2-M_p^2}{2\sqrt{|t|}}$$



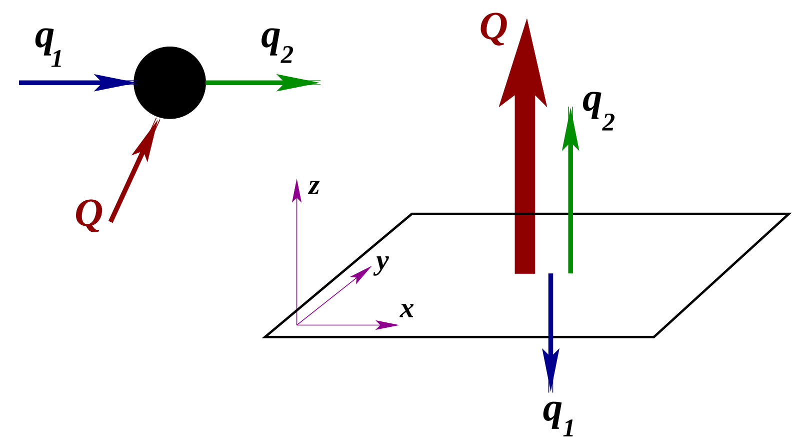
Being in the frame defined by Q and p_1 can be encoded using of δ-functions:

$$\int d^4Q\ \delta(Q^2 - t)\ \frac{d^3p_1}{2p_1^0}\ \delta^+\big(M_p^2 - (p_1 + Q)^2\big)$$
$$\delta^{(012)}\left(\frac{Q}{\sqrt{|t|}}\right)\ \delta^{(12)}\left(\frac{p_1}{p_1^0}\right)\ \frac{16}{|t|\beta(t, M_p^2, m_1^2)}\ f(p_1, Q) \equiv f(\hat{p}_1, \hat{Q}),$$

“pretending” that Q and p_1 are integration variables.

$$\beta(a, b, c) \equiv \frac{1}{2a}\sqrt{\lambda(a, b, c)},\quad \lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc.$$

Generic identity, for spacelike Q



The Breit frame is defined using generic elegant identity:

$$\int d^4Q\ \delta(Q^2-t)\ \int d^4q_1\ \delta^+(q_1^2-m_1^2)\ \int d^4q_2\ \delta^+(q_2^2-m_2^2)\ \int \delta^4(Q+q_1-q_2)$$
$$\delta^{(12)}\left(\frac{\vec{q}_1}{q_1^0}\right)\delta^{(012)}\left(\frac{Q}{\sqrt{|t|}}\right)\ \frac{16}{|t|\beta(t,m_1^2,m_2^2)}\ f(q_1,q_2,Q)\equiv f(\hat{q}_1,\hat{q}_2,\hat{Q}),$$

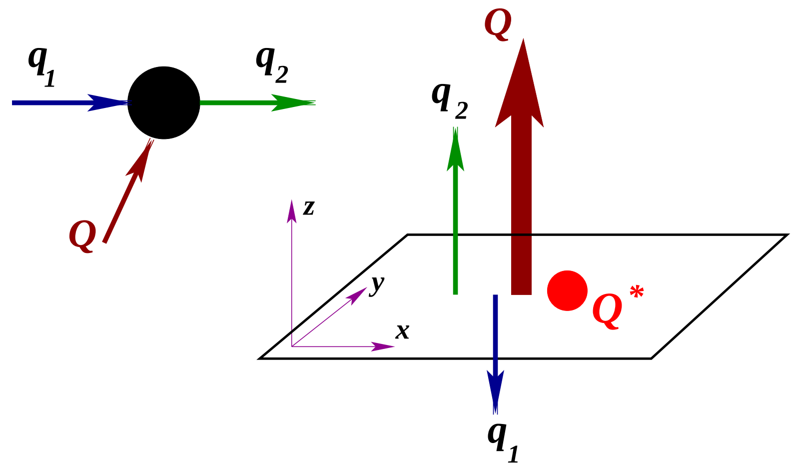
δ**'s freeze momenta to:** $\hat{Q}=(0,0,0,\sqrt{|t|})$, $\hat{q}_1=(\hat{q}_1^0,0,0,\hat{q}_1^3)$, $\hat{q}_2=(\hat{q}_2^0,0,0,\hat{q}_2^3)$,

$$\hat{q}_1^0=\hat{q}_2^0=\frac{\sqrt{|t|}}{2}\beta(t,m_1^2,m_2^2)=\frac{1}{2\sqrt{|t|}}\sqrt{\lambda(t,m_1^2,m_2^2)},$$

$\hat{q}_1^3=\frac{t+m_1^2-m_2^2}{2\sqrt{|t|}}$, $\quad \hat{q}_2^3=\frac{t-m_2^2+m_1^2}{2\sqrt{|t|}}$, $\quad \hat{q}_1^3\neq\hat{q}_2^3$**!!! Not a rest frame of** (q_1+q_2)**!**

$$\beta(a,b,c)\equiv\frac{1}{2a}\sqrt{\lambda(a,b,c)},\quad \lambda(a,b,c)\equiv a^2+b^2+c^2-2ab-2ac-2bc.$$

Breit frame of spacelike Q, using timelike $Q^\star$



It is possible to define Breit frame of spacelike Q, as a rest frame of a timelike $Q^\star$:

$$\int d^4Q\ \delta(Q^2-t)\ \int d^4q_1\ \delta^+(q_1^2-m_1^2)\ \int d^4q_2\ \delta^+(q_2^2-m_2^2)\ \int \delta^4(Q+q_1-q_2)$$
$$\delta^{(12)}\left(\frac{\vec{q}_1}{|\vec{q}_1|}\right)\delta^{(123)}\left(\frac{Q^\star}{\sqrt{|t|}}\right)\ \frac{16}{|t|\beta(t,m_1^2,m_2^2)}\ f(q_1,q_2,Q)\equiv f(\hat{q}_1,\hat{q}_2,\hat{Q}),$$

where: $Q^\star=\lambda_1 q_1+\lambda_2 q_2$, $\quad \hat{\lambda}_1=\frac{-t+m_2^2-m_1^2}{\sqrt{\lambda(t,m_1^2,m_2^2)}}$, $\quad \hat{\lambda}_2=\frac{-t-m_1^2+m_2^2}{\sqrt{\lambda(t,m_1^2,m_2^2)}}$.

By construction $\hat{q}_1,\hat{q}_2$ and $\hat{Q}$ remain the same and $\hat{Q}^\star=(\sqrt{|t|},0,0,0)$.

In short hand: $\delta^{(12)}\left(\frac{\vec{q}_1}{q_1^0}\right)\delta^{(012)}\left(\frac{Q}{\sqrt{|t|}}\right)\equiv\delta^{(12)}\left(\frac{\vec{q}_1}{|\vec{q}_1|}\right)\delta^{(123)}\left(\frac{Q^\star}{\sqrt{|t|}}\right)$

Consider: $I_n = \int dM_p^2 \int d\tau_{n+1}(\Pi_p; p_2, k_1, \dots, k_n) f(Q, p_1, p_2, k_i)$

Start in the Breit frame of $(\hat{Q}, \hat{p}_1)$**. Make** (Q, p_1) **the integration variables:**

$$I_n = \int dM_p^2 \int d^4Q \, \frac{d^3 p_1}{2p_1^0} \, \delta(Q^2 - t) \, \delta^+\left(M_p^2 - (p_1 + Q)^2\right)$$
$$\delta^{(123)}\left(\frac{Q^\star}{2\sqrt{|t|}}\right) \, \delta^{(12)}\left(\frac{\vec{p}_1}{|\vec{p}_1|}\right)$$
$$\int d\tau_{n+1}(\Pi_p; p_2, k_1, \dots, k_n) \, \frac{16}{|t|\beta(t, M_p^2, m_1^2)} \, f(Q, p_1, p_2, k_i)$$

Even more integrations and delta's are added implanting the identity:

$$\int dG(L) \, \delta^{(123)}\left(L^{-1} \frac{Q_p^\star}{\sqrt{-Q_p^2}}\right) \, \delta^{(12)}\left(L^{-1} \frac{\vec{p}_1}{|\vec{p}_1|}\right) \equiv 1, \quad Q_p = p_2 - p_1,$$

where $L = R_3(\phi) R_2(\theta) B_x$ **Lorentz transforms Breit frames** [Q, p_1] $\longrightarrow$ [Q_p, p_1].

Lorentz group integration element: $dG(L) = \sin\theta d\theta \, d\phi \, d^4x \, 2\delta^4(x^2 - 1)$.

$$I_n = \int d^4Q_p \, \delta^4(Q_p - p_2 + p_1) \int dt_p \delta(Q_p^2 - t_p)$$
$$\int dG(L) \int d^4Q \, \delta(Q^2 - t) \, \frac{d^3 p_1}{2p_1^0} \, \delta^{(123)}\left(\frac{Q^\star}{\sqrt{|t|}}\right) \, \delta^{(12)}\left(\frac{\vec{p}_1}{|\vec{p}_1|}\right)$$
$$\delta^{(123)}\left(L^{-1} \frac{Q_p^\star}{\sqrt{|t_p|}}\right) \, \delta^{(12)}\left(L^{-1} \frac{\vec{p}_1}{|\vec{p}_1|}\right)$$
$$\delta^4(Q + p_1 - p_2 - \textstyle\sum_j k_j) \, \frac{16}{|t|\beta(t, (Q+p_1)^2, m_1^2)} \, f(Q, p_1, p_2, k_i).$$

We expresesed $\Pi_p = Q + p_1$**,eliminated** $\int M_p^2 \delta(M_p^2 \dots)$**, defined** $Q_p = p_2 - p_1$.

Change of variables

Change of variables in the integral, which is just identical to the Lorentz transformation L; it formally looks as follows:

$$Q \to LQ,\ Q_p \to LQ_p,\ p_i \to Lp_i,\ k_j \to Lk_j,$$

Read: Substitute $p = Lp'$; New variables are $p' = L_A^{-1} p$; Rename $p' \to p$.

$$I_n = \int d^4Q_p\ \delta^4(Q_p - p_2 + p_1) \int dt_p \delta(Q_p^2 - t_p)$$
$$\int dG(L) \int d^4Q\ \delta(Q^2 - t)\ \frac{d^3p_1}{2p_1^0}\ \delta^{(123)}\left(L \frac{Q^\star}{\sqrt{|t|}}\right)\ \delta^{(12)}\left(L \frac{\vec{p}_1}{|\vec{p}_1|}\right)$$
$$\delta^{(123)}\left(\frac{Q_p^\star}{\sqrt{|t_p|}}\right)\ \delta^{(12)}\left(\frac{\vec{p}_1}{|\vec{p}_1|}\right)$$
$$\delta^4(Q + p_1 - p_2 - \textstyle\sum_j k_j)\ 8 \frac{1}{|t|} \beta(t, (Q + p_1)^2, m_1^2)\ f(Q, p_1, p_2, k_i).$$

We shall now reduce most of δ functionds, group by group... see next slides...

$$I_n = \int dt_p \int d^4Q_p\ \delta^4(Q_p - p_2 + p_1)\delta(Q_p^2 - t_p)$$
$$\int dG(L)\ \int d^4Q\ \delta(Q^2 - t)\ \delta^{(012)}(LQ)\ \delta^{(12)}(Lp_1)$$
$$\int \frac{d^3p_1}{2p_1^0}\ \delta^{(012)}(Q_p)\ \delta^{(12)}(p_1)\ \frac{1}{4}|t_p|^{1/2}\beta^2(t_p, m_1^2, m_2^2)$$
$$\delta^4(Q + p_1 - p_2 - \sum_j k_j)\ 4|t|\beta(t,(Q+p_1)^2, m_1^2)\ f(Q, p_1, p_2, k_i),$$

Using another variant of our “generic identity”:

$$\int d^4Q_p\ \delta^4(Q_p - p_2 + p_1)\delta(Q_p^2 - t_p)\ \int \frac{d^3p_1}{2p_1^0}\ \delta^{(012)}(Q_p)\ \delta^{(12)}(p_1)\ f(Q_p, p_1, p_2)$$
$$\equiv (4|t_p|\beta(t_p, m_1^2, m_2^2))^{-1} f(\hat{Q}_p, \hat{p}_1, \hat{p}_2)$$

we get

$$I_n = \int dt_p \int dG(L)\ \int d^4Q\ \delta(Q^2 - t)$$
$$\delta^{(012)}(LQ)\ \delta^{(12)}(L\hat{p}_1)\ \delta^4(Q + \hat{p}_1 - \hat{p}_2 - \sum_j k_j)$$
$$\frac{1}{4}|t_p|^{-1/2}\beta(t_p, m_1^2, m_2^2)\ |t|\beta(t,(Q+\hat{p}_1)^2, m_1^2)\ f(Q, \hat{p}_1, \hat{p}_2, k_i),$$

from now on $\hat{p}_2$ (and $\hat{p}_1$) are not integration variables any more!

Eliminating $\int d^4Q$ fixes $Q \to \hat{Q} = -\hat{p}_1 + \hat{p}_2 + \sum_j k_j$ giving us:

$$I_n = \int dt_p \int dG(L)\ \delta^{(012)}(L\hat{Q})\ \delta^{(12)}(L\hat{p}_1)$$
$$\frac{1}{4}|t_p|^{-1/2}\beta(t_p, m_1^2, m_2^2)\ |t|\beta(t,(\hat{p}_2 + \sum_j k_j)^2, m_1^2)$$
$$\delta((\hat{p}_1 - \hat{p}_2 - \sum_j k_j)^2 - t)\ f(Q, \hat{p}_1, \hat{p}_2, k_i).$$

Last step will be partial elimination of $\int dG(L)$...

Conclusions

- **Cascade parametrization of the ph.sp. for MC's with multibranching is most common and very succesful.**
- **Lorentz group parametrization is an interesting and very powerfull alteranative (or extension). Keep it in mind!**