

**Precision Monte Carlo KKMC for  
Fermion Pair Production at Linear Colliders  
based on  
the Coherent Exclusive Exponentiation**

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**Outline:**

- **What is the problem? Main features of  $KKMC$**
- **Kleiss-Stirling+GPS spinor technique**
- **CEEX Amplitudes and differ. cross sections**
- **Example numerical results**
- **Conclusions**

Papers by S. Jadach, B.F.L. Ward and Z. Wąs:

CERN-TH/98-253, Phys. Lett. B449 (1999) 97, CERN-TH/98-235, subm. to EPJ,

DESY-99-106 (CERN-TH/99-235), Comput. Phys. Commun. in press

**These slides are on <http://home.cern.ch/jadach>**

## The problem

Deficiencies of YFS **Exclusive Exponentiation** (EEX) technique, as implemented in KORALZ/YFS3, BHLUMI, BHWIDE, KORALW:

- Lack of the ISR/FSR or Up/Down interferences (except BHWIDE, but difficult to upgrade...)
- Simplified treatment of spin, neglected transverse polarization (except KORALB, but no exponentiation...)
- Approximate matrix element for 2 and 3 hard large  $p_T$  photons. (Important for searches.)

## The CEEEX solution

- ISR/FSR interferences are included in a natural way. Spin amplitudes for ISR and FSR summed squared numerically. (The BHWIDE approach gets cumbersome beyond first order)
- Exact treatment of fermion spin polarizations (transv.&longitudinal). Numerical Wigner rotations of spin density matrices available. (Necessary for interfacing with decay M.C. simulating fermion decays).
- Exact matrix element for 2 and 3 and more photons using Kleiss-Stirling (KS) spinor technique.

# INTRODUCTION

I-3

## Comparison of $\mathcal{K}\mathcal{K}$ MC with its predecessors

Feature	KORALB	KORALZ	$\mathcal{K}\mathcal{K}$ 4.13	$\mathcal{K}\mathcal{K}$ 2000+?
QED type	$\mathcal{O}(\alpha)$	EEX	CEEX, EEX	CEEX, EEX
CEEX(ISR+FSR)	none	none	$\{\alpha, \alpha L; \alpha^2 L^2, \alpha^2 L^1\}$	$\{\dots \alpha^2 L^1; \alpha^3 L^3\}$
EEX(ISR*FSR)	none	$\{\alpha, \alpha L, \alpha^2 L^2\}$	$\{\alpha, \alpha L, \alpha^2 L^2, \alpha^3 L^3\}$	$\{\dots \alpha^2 L^2, \alpha^3 L^3\}$
ISR-FSR int.	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha)$	$\{\alpha, \alpha L\}_{\text{CEEX}}$	$\{\alpha, \alpha L\}_{\text{CEEX}}$
Exact brems.	1 $\gamma$	1, 2coll. $\gamma$	1, 2, 3coll. $\gamma$	up to 3 $\gamma$
Electroweak	No Z-res.	DIZET 6.x	DIZET 6.x	New version?
Beam polar.	long+trans.	longit.	long+trans.	long+trans.
$\mathcal{T}$ polar.	long+trans.	longit.	long+trans.	long+trans.
Hadronization	—	JETSET	JETSET	PYTHIA
$\mathcal{T}$ decay	TAUOLA	TAUOLA	TAUOLA	TAUOLA
Inclusive mode	—	No	Yes	Yes
Beamstrahlung	—	No	Yes	Yes
Beam spread	—	No	Yes	Yes
$\nu\nu$ channel	—	Yes	No	Yes
$ee$ channel	—	No	No	Yes
$tt$ channel	—	No	No	yes?
$WW$ channel	—	No	No	yes?

## Notorious Bhabha: list of the leading problems

- Going beyond  $\mathcal{O}(\alpha^1)$  mandatory, precision  $< 0.5\%$ .  
Hence, up to four particles in the final state, and two loop virtual corrections.
- Monte Carlo is mandatory: Why?
  1. 4-body final states,
  2. nontrivial experimental cuts,
  3. calorimetric measurements of electrons and photons.
- Exponentiation is mandatory: Why?
  1. To have realistic MC multi-photons, as in experiment.
  2. To avoid negative cross sections. In  $\mathcal{O}(\alpha^1)$  the minimum real photon energy  $k_0$  cannot be small enough because MC weight (diff. xsect.) becomes negative, (events with 0-real and 1-virtual photon). Moreover keeping  $k_0 > 0.005$  causes technical error  $\sim 0.5\%$ . Going to  $\mathcal{O}(\alpha^2)$  without expon. does not help. Weights  $< 0$  (diff. xsect.  $< 0$ ) still there, now for events with 1-real and 1-virtual photon. (Surprise to some people during 1990 LEP workshop).
  3. Construction of complete  $\mathcal{O}(\alpha^2)$  MC will take time, it is desirable to split it into stages. If exponentiation adopted then one deals with the IR-finite building-blocks, it is therefore possible to have an intermediate working solution with the incomplete  $\mathcal{O}(\alpha^2)$ , for example without sub-sub-leading terms, or without difficult virtual corrections, before available.
  4. Solutions based on the leading-logs (LL) combined with  $\mathcal{O}(\alpha^1)$  are plagued with the problem of the LL-scale. Within LL there is no way to say whether one use  $s$  or  $t$ -scale in the LL QED correction. In reality transition from  $s$  to  $t$  scale occur smoothly when going from  $s$  to  $t$  angles. LL ignores this fact. Another problem: the LL scale is not defined for  $\gamma - Z$  interference where  $Z$  is  $s$ -channel resonance and gamma is  $t$ -channel exchange.

## Light in the tunnel!

Why? The phase space integration of 2 and more photons, done routinely since long, without any approximation as in BHLUMI and KORALZ and KKMC. The only problem is with matrix element, and here an important progress: CEEEX has many features needed to solve the  $\mathcal{O}(\alpha^2)$  Bhabha problem:

1. No problem with 2-photon 4-body phase space.
2. No problem with negative cross sections.
3. No problems with the LL scale. Controlled automatically by the constructive/destructive interferences. Smooth transition between s to t-scales, when going to small angles.
4. No problem with Z resonance and its interference with the rest,  $\Gamma/M$  cancellations under control.
5. No problem with dividing development of the project into stages. All virtual corrections enter as IR-free objects, can be added one by one as soon as they are available. (Penta-boxes etc., 2-loop boxes etc.)
6. Exact  $2\gamma$  emission amplitudes using spinor methods.
7. Electroweak  $\mathcal{O}(\alpha^1)$  corrs. enter consistently/naturally.
8. Beam polarizations implemented trivially.

## Various past efforts:

**BABAMC:** True  $\mathcal{O}(\alpha^1)$  Monte Carlo event generator. (Double tag. All angles.) Authors were doing enormous effort (see paper) to fight the so called  $k_0$  problem (lack of exponentiation).

**ALIBABA:** MC integrator, no event simulation.  $\mathcal{O}(\alpha^1)$  combined with LL. (Double tag. All angles.) Problem with LL scale. Certain heuristic choice was done to deal with it, but it was never fully satisfactory nor justified.

**BHLUMI:** True MC event simulation. It is full  $\mathcal{O}(\alpha^1) + \mathcal{O}(\alpha^2)$ LL,  $\mathcal{O}(\alpha^3)$ LL Restricted to small t, therefore no problem with LL scale. Extension of large angles practically impossible.

**BHWIDE:** True MC event simulation. Full  $\mathcal{O}(\alpha^1)$  with YFS exponentiation. Exponentiation method strongly limited to  $\mathcal{O}(\alpha^1)$ .

**BHAGEN of Bologna:** Hybrid MC event generator, By now complete  $\mathcal{O}(\alpha^1) + \text{LL}$ , problems with technical precision. Upgrade to full second order declared impossible. In particular problems with 2-photon phase space.

**SABSPV of Pavia:** Hybrid MC event generator, It relies too strongly on LL Structure functions Probably will be never able to include true second order NLL. Problem with LL scale. Problems with 2-photon phase space.

## Kleiss-Stirling spinors

Any MASSLESS spinors is transformed out of two

**basic constant** spinors  $u_{\pm}(\zeta)$ ,  $\zeta^2 = 0$ :  $u_{\lambda}(p) = \frac{1}{\sqrt{2p \cdot \zeta}} \not{p} u_{-\lambda}(\zeta)$ ,

where  $u_{+}(\zeta) = \not{\eta} u_{-}(\zeta)$ ,  $\eta^2 = -1$ ,  $(\eta\zeta) = 0$ .

$\not{\zeta} u_{\lambda}(\zeta) = 0$ ,  $\omega_{\lambda} u_{\lambda}(\zeta) = u_{\lambda}(\zeta)$ ,  $u_{\lambda}(\zeta) \bar{u}_{\lambda}(\zeta) = \not{\zeta} \omega_{\lambda}$ ,

$\not{p} u_{\lambda}(p) = 0$ ,  $\omega_{\lambda} u_{\lambda}(p) = u_{\lambda}(p)$ ,  $u_{\lambda}(p) \bar{u}_{\lambda}(p) = \not{p} \omega_{\lambda}$ , and  $\omega_{\lambda} = \frac{1}{2}(1 + \lambda\gamma_5)$ .

Spinors for the MASSIVE hal spin fermion with 4-momentum  $p$  ( $p^2 = m^2$ ) and spin projection  $\lambda/2$  are defined in terms of massless spinors:

$$u(p, \lambda) = u_{\lambda}(p_{\zeta}) + \frac{m}{\sqrt{2p_{\zeta}}} u_{-\lambda}(\zeta) \quad \text{and} \quad v(p, \lambda) = u_{-\lambda}(p_{\zeta}) - \frac{m}{\sqrt{2p_{\zeta}}} u_{\lambda}(\zeta),$$

where  $p_{i\zeta} \equiv \hat{p}_i \equiv p_i - \zeta m_i^2 / (2\zeta p_i)$ .

We exploit very often the standard completeness relations:

$$\not{p} + m = \sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda), \quad \not{p} - m = \sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda)$$

## Toolbox for Kleiss-Stirling spinors

The **inner product** of the two **massless** spinors is defined as follows:

$$s_+(p_1, p_2) \equiv \bar{u}_+(p_1)u_-(p_2), \quad s_-(p_1, p_2) \equiv \bar{u}_-(p_1)u_+(p_2) = -(s_+(p_1, p_2))^*.$$

In any reference frame it can be evaluated using the Kleiss-Stirling expression:

$$s_+(p, q) = 2 (2p\zeta)^{-1/2} (2q\zeta)^{-1/2} [(p\zeta)(q\eta) - (p\eta)(q\zeta) - i\epsilon_{\mu\nu\rho\sigma}\zeta^\mu\eta^\nu p^\rho q^\sigma]$$

For example, if in LAB frame  $\zeta = (1, 1, 0, 0)$  and  $\eta = (0, 0, 1, 0)$ , then in this frame:

$$s_+(p, q) = -(q^2 + iq^3)\sqrt{(p^0 - p^1)/(q^0 - q^1)} + (p^2 + ip^3)\sqrt{(q^0 - q^1)/(p^0 - p^1)}.$$

The **inner product** of the two **massive** spinors is:

$$\bar{u}(p_1, \lambda_1)u(p_2, \lambda_2) = S(p_1, m_1, \lambda_1, p_2, m_2, \lambda_2),$$

$$\bar{u}(p_1, \lambda_1)v(p_2, \lambda_2) = S(p_1, m_1, \lambda_1, p_2, -m_2, -\lambda_2),$$

$$\bar{v}(p_1, \lambda_1)u(p_2, \lambda_2) = S(p_1, -m_1, -\lambda_1, p_2, m_2, \lambda_2),$$

$$\bar{v}(p_1, \lambda_1)v(p_2, \lambda_2) = S(p_1, -m_1, -\lambda_1, p_2, -m_2, -\lambda_2),$$

where

$$S(p_1, m_1, \lambda_1, p_2, m_2, \lambda_2) = \delta_{\lambda_1, -\lambda_2} s_{\lambda_1}(p_1\zeta, p_2\zeta) + \delta_{\lambda_1, \lambda_2} \left( m_1 \sqrt{\frac{2\zeta p_2}{2\zeta p_1}} + m_2 \sqrt{\frac{2\zeta p_1}{2\zeta p_2}} \right)$$

and  $p_{i\zeta} \equiv \hat{p}_i \equiv p_i - \zeta m_i^2 / (2\zeta p_i)$



## Why do we need GPS?

**GOODIES** of the Wigner-Jacob-Wick spinology we know and love:

**Elegant definition of spin state and scattering matrix element**

$$|p, n \rangle = T(L(p)) |\hat{p}, n \rangle, \quad \mathcal{M}_{nm} = \langle f|S|i \rangle = \langle q, m|S|p, n \rangle$$

**Simple transformation under rotation using Wigner  $\mathcal{D}^s$ -matrices**

$$T(R) |\hat{p}, n \rangle = \sum_{n'} |\hat{p}, n' \rangle \mathcal{D}_{n'n}^s(R)$$

**Well defined (so called) Wigner-Wick rotation**

$$\mathcal{M}_{nm} = \sum_{m', n'} \mathcal{M}_{n'm'} \mathcal{D}_{n'n}^s(R_i^W) \mathcal{D}_{m'm}^{s+}(R_f^W)$$

where  $R^W$  is rotation reflecting directly the change of the quantization axes (in rest frame of the massive particle) or is “generated” by the general Lorentz transformation, as in the paper of Wick (Ann. Phys. 18 (1962) 65) for the Jacob-Wick helicity states.

**How to combine the above with Weyl spinor techniques? → GPS rules.**

**WHY BOTHER?** For unstable fermions like  $\tau$  production and decay spin amplitudes are calculated using completely different methods and with different quantization axes. We need to know very precisely the spin quantization axes for KS spinors in the M.C. event generation, especially for unstable fermions.

## GPS – preliminaria

The necessary condition for validity of the usual Wigner-Jacob-Wick technology (including standard Clebsch-Gordan coefficients etc.) is the textbook phase relation:

$$(J_x \pm iJ_y)|m \rangle = [(s \mp m)(s \pm m + 1)]^{1/2} |\pm m \rangle$$

(National Bureau of Standards 1951)

equivalent to a condition with the prosaic rotation by  $+\pi$  around  $y$ -axis

$$\exp(-\frac{1}{2} i\pi J_y)|+ \rangle = |- \rangle, \text{ in our } s = 1/2 \text{ case.}$$

Take Weyl representation. Let us define Primary Reference Frame **PRF** where  $\zeta = (1, 0, 0, -1)$  and  $\eta = (0, 1, 0, 0)$ , (in general  $\text{PRF} \neq \text{LAB}$ ). If the massive fermion is **at rest** in the PRF then for KS spinors  $u(p, \pm)$  and  $v(p, \pm)$  the above phase relation is fulfilled, i.e., spin is quantized using  $z$ -axis; the  $x$  and  $y$  axes are correctly positioned!

In the general case of the fermion **in flight** the GPS rule will tell us where the quantization  $z$ -axis is, and also the  $x$  and  $y$  axes relevant for the relative phases of  $|\pm \rangle$ . See next slide.

## GPS stands for Global Positioning of Spin

The rules for determining all three spin quantization axes for  $u(p, \pm)$  and  $v(p, \pm)$  defined with KS method for Weyl representation are the following:

- In the rest frame of the fermion, take the  $z$ -axis along  $-\vec{\zeta}$ .
- Place the  $x$ -axis in the plane defined by the  $z$ -axis from the previous point and the vector  $\vec{\eta}$ , in the same half-plane as  $\vec{\eta}$ .
- With the  $y$ -axis, complete the right-handed system of coordinates.

We call the above rules the GPS rules, and we shall call the Spin Quantization Reference Frame (SQRF) determined by the above GPS rules the **GPS frame** of the fermion.

The formal proofs of the above rules is in CERN-TH-98-235. It amounts to showing that in GPS fermion rest frame the two *constant basic* spinors  $u_{\pm}(\zeta)$ , up to a real constant, are the same (have the same components) as in PRF.

## GPS at work: polarized $\tau$ decay

Suppose, that for  $\tau(p) \rightarrow X$  decay, using Feynman rules and classic “Dirac alchemy”, including  $u(p, s) \bar{u}(p, s) = \frac{1}{2}(1 + \gamma_5 \not{s})(\not{p} + m)$ , we obtain (def. polarimeter vector  $h$ ):

$$d\Gamma_{class.}^{pol.}(s) = d\Gamma^{unpol.}(q_1 \dots q_n) (1 + s \cdot h(q_1 \dots q_n)), \text{ where } h \cdot p = 0.$$

On the other hand, using KS spinors, we calculate spin amplitudes  $\mathcal{N}_\mu$  for decay process.

Remembering the textbook definition of spin density matrix we can write:

$$d\Gamma_{KS}^{pol.}(s) = \sum_{\mu, \bar{\mu}=\pm 1/2} \rho_{\mu\bar{\mu}} \mathcal{N}_\mu \mathcal{N}_{\bar{\mu}}^* d\Phi_{dec.} \text{ where } \rho \text{ is the spin density matrix.}$$

**HOW TO RELATE  $\mathcal{N}_\mu$  AND  $h_\mu$  IN THE TWO CALCULATION METHODS?**

Seems easy: from textbook relation  $\rho = \sum_{k=0}^3 \sigma^k \hat{s}^k$  where  $\hat{s} = (1, \vec{s})$  we get:

$$d\Gamma^{pol.}(s) = \hat{s}^a \left[ \sigma_{\mu\bar{\mu}}^a \mathcal{N}_\mu \mathcal{N}_{\bar{\mu}}^* \right] d\Phi_{dec.} = \hat{s}^a \hat{h}_a d\Gamma^{unpol.} \text{ and therefore we IDENTIFY:}$$

$$\hat{h}_a = (1, \vec{h}(q_1 \dots q_n)) = \sum_{\rho\bar{\rho}} \sigma_{\rho\bar{\rho}}^a \mathcal{N}_\rho \mathcal{N}_{\bar{\rho}}^* / \left( \sum_{\rho} \mathcal{N}_\rho \mathcal{N}_{\rho}^* \right) \text{ where } \sigma^a \text{ are Pauli}$$

matrices. All the above was in the  $\tau$  rest frame. **WHICH FRAME?**

**ONLY in the GPS rest frame! Why? In other frame we would be forced to replace Pauli matrices with something**

**else;** i.e. with non-standard analog of Clebsch-Gordan coefficients, because the phase relation  $(J_x + iJ_y)|-\rangle = |+\rangle$  holds for our spin

states only in the GPS frame (note that Pauli  $\sigma$ 's in  $\rho = \sum_{k=0}^3 \sigma^k \hat{s}^k$  are here just Clebsch-Gordan coefficients for  $D^{1/2} \otimes D^{1/2}$ ).

## GPS at work: Wigner/Wick rotation

For the  $\tau$  production and decay process

$$e^-(p_1, \lambda_1) + e^+(p_2, \lambda_2) \rightarrow \tau^-(q_1, \mu_1) + \tau^+(q_2, \mu_2), \tau^\pm \rightarrow X^\pm,$$

following the same lines, we may write the spin amplitudes:

$$\mathcal{M}_{\lambda_1 \lambda_2} = \sum_{\mu_1 \mu_2} \mathcal{M}_{\lambda_1 \lambda_2 \mu_1 \mu_2} \mathcal{N}_{\mu_1} \mathcal{N}'_{\mu_2}$$

and the polarized differential cross-section:

$$d\sigma = \sum_{\mu_i \bar{\mu}_i \lambda_i \bar{\lambda}_i} \rho_{\lambda_1 \bar{\lambda}_1} \rho'_{\lambda_2 \bar{\lambda}_2} \mathcal{M}_{\lambda_1 \lambda_2 \mu_1 \mu_2} \mathcal{M}_{\bar{\lambda}_1 \bar{\lambda}_2 \bar{\mu}_1 \bar{\mu}_2}^* d\Phi_{\text{prod.}} \mathcal{N}_{\mu_1} \mathcal{N}_{\bar{\mu}_1}^* d\Phi_\tau \mathcal{N}'_{\mu_2} \mathcal{N}'_{\bar{\mu}_2}^* d\Phi'_\tau$$

In terms of beam polarizations  $\hat{s}$  and decay polarimeter vectors  $\hat{h}$  we have:

$$d\sigma = \sum_{abcd} \hat{s}^a \hat{s}'^b R_{ab}^{cd} d\sigma_{\text{unpol.}}^{\text{prod.}} \hat{h}_c d\Gamma_{\text{unpol.}} \hat{h}'_d d\Gamma'_{\text{unpol.}} \quad \text{where}$$

$$R_{ab}^{cd} = \frac{\sum_{\mu_i \bar{\mu}_i \lambda_i \bar{\lambda}_i} \sigma_{\lambda_1 \bar{\lambda}_1}^a \sigma_{\lambda_2 \bar{\lambda}_2}^b \mathcal{M}_{\lambda_1 \lambda_2 \mu_1 \mu_2} \mathcal{M}_{\bar{\lambda}_1 \bar{\lambda}_2 \bar{\mu}_1 \bar{\mu}_2}^* \sigma_{\bar{\mu}_1 \mu_1}^c \sigma_{\bar{\mu}_2 \mu_2}^d}{\sum_{\mu_i \bar{\mu}_i \lambda_i \bar{\lambda}_i} |\mathcal{M}_{\lambda_1 \lambda_2 \mu_1 \mu_2}|^2}$$

The above Wigner-Jacob-Wick spin technology is used in KORALB MC.

If the production amplitudes  $\mathcal{M}_{\lambda_1 \lambda_2 \mu_1 \mu_2}$  are calculated using KS spinors then we may continue to use the above “spinology” **only if we apply it in the GPS** rest frames of all four fermions! **Consequently, we need Wigner/Wick rotations. Why? See next slide.**

## GPS at work: Wigner/Wick rotation

For  $e^\pm$  beams we know polarization vectors  $\vec{s}_\pm$  in certain well defined rest frames  $\text{BRF}_\pm$  of the  $e^\pm$  (from the machine people). In other words, we know precisely the Lorentz transformations  $L_\pm^{\text{BRF}}: \text{BRF}_\pm \longrightarrow \text{LAB}$ . On the other hand, from GPS

rules, we know transformations  $L_\pm^{\text{GPS}}: \text{GPS}(e^\pm) \longrightarrow \text{LAB}$ . If we use KS spinors for the scattering spin amplitudes, then we have to apply the Wigner/Wick rotation

$R_\pm^W = L_\pm^{\text{GPS}}(L_\pm^{\text{BRF}})^{-1}$ , i.e. rotate  $\vec{s}_\pm$  from  $\text{BRF}_\pm$  to  $\text{GPS}(e^\pm)$ . (In practice this is a rotation around the beam axis).

For  $\tau^\pm$  we avoid explicit Wigner/Wick rotation, if the decays of polarized  $\tau$ 's are done (simulated in MC) precisely in  $\text{GPS}(\tau^\pm)$  frames.

We may need Wigner/Wick rotation if we want to compare spin amplitudes from KS method and some other method. Next slide shows such a comparison for the spin correlation tensor  $R_{00}^{cd}$  from two methods: (1) KS spinors, (2) with Jacob-Wick (JW) helicity states (Jadach Was 1984, Tsai 1971); in fact the variant of JW called JW2.



## Appendix A: Wigner-Wick rotations, collection of formulae

**STATES** in Hilbert space transform under rotation as follows:

$$|\overset{\circ}{p}, \mu \rangle_R = \sum_{\mu'} |\overset{\circ}{p}, \mu' \rangle \mathcal{D}_{\mu' \mu}^{1/2}(R), \quad \langle \overset{\circ}{p}, \mu |_R = \sum_{\mu'} \mathcal{D}_{\mu \mu'}^{1/2 \dagger}(R) \langle \overset{\circ}{p}, \mu' |.$$

Consequently, **SPIN AMPLITUDES** transform under four different Wigner rotations:

$$\mathcal{M}_{\lambda_1 \lambda_2 \mu_1 \mu_2} = \sum_{\lambda_i \mu_i} \mathcal{M}_{\lambda'_1 \lambda'_2 \mu'_1 \mu'_2} \mathcal{D}_{\lambda'_1 \lambda_1}^{1/2}(R_1^i) \mathcal{D}_{\lambda'_2 \lambda_2}^{1/2}(R_2^i) \mathcal{D}_{\mu_1 \mu'_1}^{1/2 \dagger}(R_1^f) \mathcal{D}_{\mu_2 \mu'_2}^{1/2 \dagger}(R_2^f),$$

Final an initial state spin **DENSITY MATRICES** transform by Wigner rotation in the same way:

$$(\rho_{\mu \bar{\mu}})_R = \sum_{\mu' \bar{\mu}'} \mathcal{D}_{\mu \mu'}^{1/2 \dagger}(R) \rho_{\mu' \bar{\mu}'} \mathcal{D}_{\bar{\mu}' \bar{\mu}}^{1/2}(R)$$

The above transformation induces through  $s^k = \text{Tr}(\rho \sigma^k)$  the ordinary rotation

transformation of the spin **POLARIZATION VECTOR** (as it should!) e.g.  $\vec{s}$  transforms as a contravariant vector:

$$(s^k)_R = \sum_{k'=1}^3 R^k_{k'} s^{k'},$$

On the other hand the  $h_a$  **POLARIMETER VECTOR** (also related to spin amplitudes with Pauli matrices) transforms with transposed rotation matrix, e.g. as a covariant vector:

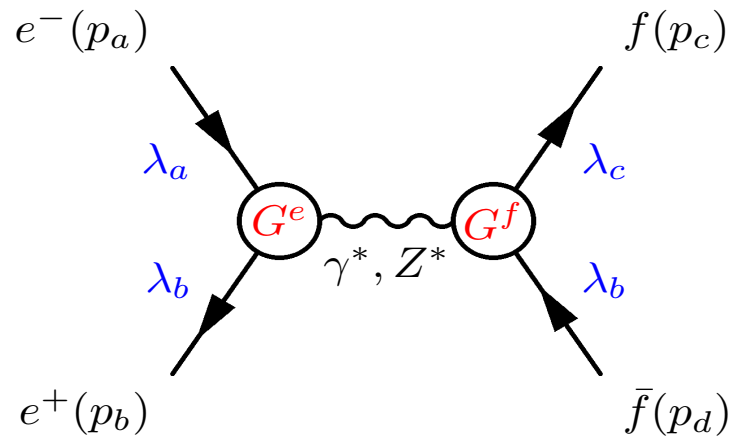
$$(h_a)_R = \sum_{a'=1}^3 h_{a'} R^{a'}_a$$

Finally the full spin **CORRELATION TENSOR** transforms as follows:

$$(R_{ab}^{cd})_R = \sum_{a' b' c' d'=1}^3 R_{a' b'}^{c' d'} (R_1^i)^{a'}_a (R_2^i)^{b'}_b (R_1^f)^c_{c'} (R_2^f)^d_{d'}. \quad \text{Easy?!}$$

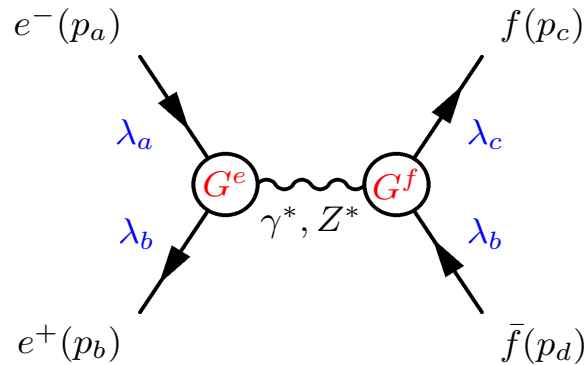


## GPS Appendix B: Born Spin Amplitudes, definition



$$\begin{aligned} \mathfrak{B} \left( \begin{matrix} p \\ \lambda \end{matrix}; X \right) &= \mathfrak{B} \left( \begin{matrix} p_a & p_b & p_c & p_d \\ \lambda_a & \lambda_b & \lambda_c & \lambda_d \end{matrix}; X \right) = \\ &= 2ie^2 \sum_{B=\gamma, Z} \frac{\bar{v}(p_b, \lambda_b) \gamma^\mu G^{e,B} u(p_a, \lambda_a) \bar{u}(p_c, \lambda_c) \gamma_\mu G^{f,B} v(p_d, \lambda_d)}{X^2 - M_B^2 + i\Gamma_B X^2 / M_B}, \\ G^{e,B} &= \sum_{\lambda=\pm} \frac{1}{2} (1 + \lambda \gamma_5) g_\lambda^{e,B}, \quad G^{f,B} = \sum_{\lambda=\pm} \frac{1}{2} (1 + \lambda \gamma_5) g_\lambda^{f,B}, \end{aligned}$$

## GPS Appendix C: Born Spin Amplitudes in terms of KS spinors



$$\begin{aligned} \mathfrak{B} \left( \begin{matrix} p_a & p_b & p_c & p_d \\ \lambda_a & \lambda_b & \lambda_c & \lambda_d \end{matrix}; X \right) &= ie^2 \sum_{B=\gamma, Z} \frac{\bar{v}(p_b, \lambda_b) \gamma^\mu G^{e,B} u(p_a, \lambda_a) \bar{u}(p_c, \lambda_c) \gamma_\mu G^{f,B} v(p_d, \lambda_d)}{X^2 - M_B^2 + i\Gamma_B X^2/M_B} \\ &= 2ie^2 \sum_{B=\gamma, Z} \frac{\delta_{\lambda_a, -\lambda_b} \left[ g_{\lambda_a}^{e,B} g_{-\lambda_a}^{f,B} T_{\lambda_c \lambda_a} T'_{\lambda_b \lambda_d} + g_{\lambda_a}^{e,B} g_{\lambda_a}^{f,B} U'_{\lambda_c \lambda_b} U_{\lambda_a \lambda_d} \right]}{X^2 - M_B^2 + i\Gamma_B X^2/M_B}, \\ T_{\lambda_c \lambda_a} &= \bar{u}(p_c, \lambda_c) u(p_a, \lambda_a) = S(p_c, m_c, \lambda_c, p_a, 0, \lambda_a), \\ T'_{\lambda_b \lambda_d} &= \bar{v}(p_b, \lambda_b) v(p_d, \lambda_d) = S(p_b, 0, -\lambda_b, p_d, -m_d, -\lambda_d), \\ U'_{\lambda_c \lambda_b} &= \bar{u}(p_c, \lambda_c) v(p_b, -\lambda_b) = S(p_c, m_c, \lambda_c, p_b, 0, \lambda_b), \\ U_{\lambda_a \lambda_d} &= \bar{u}(p_a, -\lambda_a) v(p_d, \lambda_d) = S(p_a, 0, -\lambda_a, p_d, -m_d, -\lambda_d). \end{aligned}$$

## Conclusions for Part II

- Our primary aim is to use KS spinors for multiple bremsstrahlung spin amplitudes, for massive fermions.
- Additional important aims is to handle (Wigner rotations) efficiently and without any approximations, the full spin polarization vectors (= dens. matr.) for any fermion — very important for interfacing MC simulation of production and decay of unstable fermions like  $\tau$  and top quark.
- With the GPS upgrade of the KS technique we are fully armed to implement completely the above ambitious scenario!

## Outline of Part II

- Preliminaria: Photon polarization vector,  
Building blocks for bremsstrahlung amplitudes: U and V matrices.
- 1-photon real using KS, isolate IR and non-IR parts, notation,  
and 1-photon virtual, collect old results.
- **Warm up: CEEX, zero order  $\mathcal{O}(\alpha^0)_{\text{CEEX}}$ .**
- **Full scale:  $\mathcal{O}(\alpha^r)_{\text{CEEX}}$ ,  $r = 1, 2$ .**
- Spin structure, connection to decays of final fermion.
- IR cancellations, IR-cut for real photons, etc.
- CPU time considerations and photon spin randomization.
- Appendix on first and second order  $\beta$ 's.

## Photon polarization vector

For a circularly polarized photon with four-momentum  $k$  and helicity  $\sigma = \pm 1$  we adopt the choice of Kleiss-Stirling and/or Beijing group:

$$(\epsilon_{\sigma}^{\mu}(k, \beta))^* = \frac{\bar{u}_{\sigma}(k)\gamma^{\mu}u_{\sigma}(\beta)}{\sqrt{2}\bar{u}_{-\sigma}(k)u_{\sigma}(\beta)}, \quad (\epsilon_{\sigma}^{\mu}(k, \zeta))^* = \frac{\bar{u}_{\sigma}(k)\gamma^{\mu}\mathbf{u}_{\sigma}(\zeta)}{\sqrt{2}\bar{u}_{-\sigma}(k)\mathbf{u}_{\sigma}(\zeta)},$$

where  $\beta$  is an arbitrary light-like four-vector  $\beta^2 = 0$  (axial gauge).

The second choice with  $\mathbf{u}_{\sigma}(k, \zeta)$  seem to be ours (not exploited by KS).

Using the Chisholm identity:

$$\bar{u}_{\sigma}(k)\gamma_{\mu}u_{\sigma}(\beta)\gamma^{\mu} = 2u_{\sigma}(\beta)\bar{u}_{\sigma}(k) + 2u_{-\sigma}(k)\bar{u}_{-\sigma}(\beta),$$

$$\bar{u}_{\sigma}(k)\gamma_{\mu}\mathbf{u}_{\sigma}(\zeta)\gamma^{\mu} = 2\mathbf{u}_{\sigma}(\zeta)\bar{u}_{\sigma}(k) - 2u_{-\sigma}(k)\bar{\mathbf{u}}_{-\sigma}(\zeta)$$

we get useful equivalent expressions:

$$(\not{\epsilon}_{\sigma}(k, \beta))^* = \frac{\sqrt{2}[u_{\sigma}(\beta)\bar{u}_{\sigma}(k) + u_{-\sigma}(k)\bar{u}_{-\sigma}(\beta)]}{\bar{u}_{-\sigma}(k)u_{\sigma}(\beta)},$$

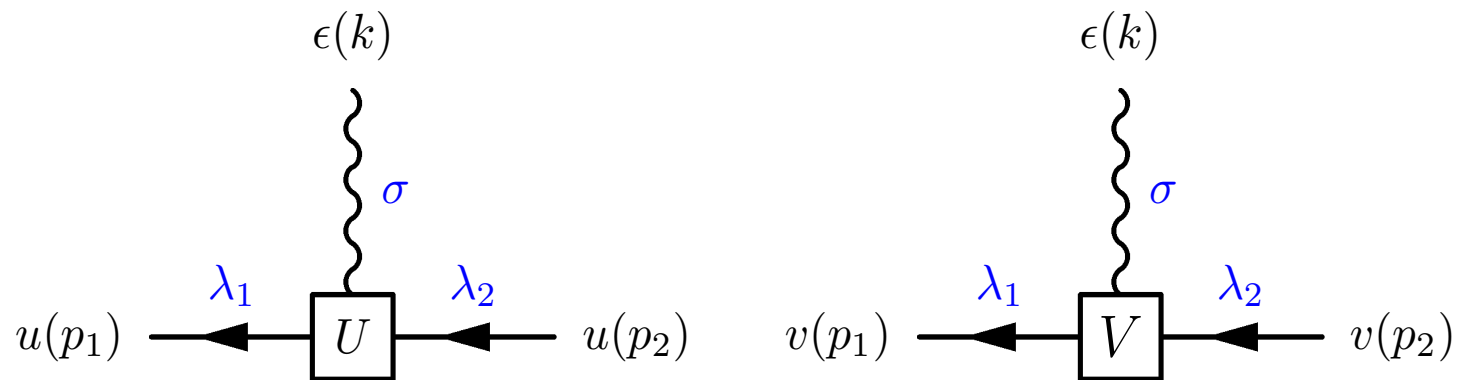
$$(\not{\epsilon}_{\sigma}(k, \zeta))^* = \frac{\sqrt{2}[\mathbf{u}_{\sigma}(\zeta)\bar{u}_{\sigma}(k) - u_{-\sigma}(k)\bar{\mathbf{u}}_{-\sigma}(\zeta)]}{\sqrt{2\zeta k}}.$$

# CEEX Amplitudes

III-3

**Building blocks for bremsstrahlung amplitudes: U and V matrices.**

Basic building blocks in our calculation of the bremsstrahlung amplitudes are:



$$U \begin{bmatrix} p_1 & k & p_2 \\ \lambda_1 & \sigma & \lambda_2 \end{bmatrix} = U_{\lambda_1, \lambda_2}^{\sigma} (k, p_1, m_1, p_2, m_2) = \bar{u}(p_1, \lambda_1) \not{\epsilon}_{\sigma}^*(k) u(p_2, \lambda_2),$$

$$V \begin{bmatrix} p_1 & k & p_2 \\ \lambda_1 & \sigma & \lambda_2 \end{bmatrix} = V_{\lambda_1, \lambda_2}^{\sigma} (k, p_1, m_1, p_2, m_2) = \bar{v}(p_1, \lambda_1) \not{\epsilon}_{\sigma}^*(k) v(p_2, \lambda_2).$$

The four-momentum conservation is not assumed in the above vertex-like objects.

## U and V transition matrices, programmable formulas

In the case of  $\epsilon_\sigma(k, \zeta)$  the transition matrices are rather simple:

$$U^+(k, p_1, m_1, p_2, m_2) = \sqrt{2} \begin{bmatrix} \sqrt{\frac{2\zeta p_2}{2\zeta k}} s_+(k, \hat{p}_1), & 0 \\ m_2 \sqrt{\frac{2\zeta p_1}{2\zeta p_2}} - m_1 \sqrt{\frac{2\zeta p_2}{2\zeta p_1}}, & \sqrt{\frac{2\zeta p_1}{2\zeta k}} s_+(k, \hat{p}_2) \end{bmatrix}$$

$$U_{\lambda_1, \lambda_2}^-(k, p_1, m_1, p_2, m_2) = \left[ -U_{\lambda_2, \lambda_1}^+(k, p_2, m_2, p_1, m_1) \right]^*,$$

$$V_{\lambda_1, \lambda_2}^\sigma(k, p_1, m_1, p_2, m_2) = U_{-\lambda_1, -\lambda_2}^\sigma(k, p_1, -m_1, p_2, -m_2).$$

The above expressions is optimized for fast numerical evaluation.

The following general case, with  $\epsilon_\sigma(k, \beta)$ , looks a little bit more complicated:

$$U^+(k, p_1, m_1, p_2, m_2) = \frac{\sqrt{2}}{s_-(k, \beta)} \times \begin{bmatrix} s_+(\hat{p}_1, k) s_-(\beta, \hat{p}_2) + m_1 m_2 \sqrt{\frac{2\zeta \beta}{2\zeta p_1} \frac{2\zeta k}{2\zeta p_2}}, & m_1 \sqrt{\frac{2\zeta \beta}{2\zeta p_1}} s_+(k, \hat{p}_2) + m_2 \sqrt{\frac{2\zeta \beta}{2\zeta p_2}} s_+(\hat{p}_1, k) \\ m_1 \sqrt{\frac{2\zeta k}{2\zeta p_1}} s_-(\beta, \hat{p}_2) + m_2 \sqrt{\frac{2\zeta k}{2\zeta p_2}} s_-(\hat{p}_1, \beta), & s_-(\hat{p}_1, \beta) s_+(k, \hat{p}_2) + m_1 m_2 \sqrt{\frac{2\zeta \beta}{2\zeta p_1} \frac{2\zeta k}{2\zeta p_2}} \end{bmatrix}$$

NB. They are not  $M$ -matrices of KS, but rather products of them.

## Diagonality property of U and V

For  $p_1 = p_2 = p$  the matrices  $U$  and  $V$  become diagonal:

$$U \begin{bmatrix} p & kp \\ \lambda_1 \sigma \lambda_2 \end{bmatrix} = V \begin{bmatrix} p & kp \\ \lambda_1 \sigma \lambda_2 \end{bmatrix} = b_\sigma(k, p) \delta_{\lambda_1 \lambda_2},$$

$$b_\sigma(k, p) = \sqrt{2} \frac{\bar{u}_\sigma(k) \not{p} \mathbf{u}_\sigma(\zeta)}{\bar{u}_{-\sigma}(k) \mathbf{u}_\sigma(\zeta)} = \sqrt{2} \sqrt{\frac{2\zeta p}{2\zeta k}} s_\sigma(k, \hat{p}).$$

In a more general case of  $\epsilon(k, \beta)$  diagonality also holds:

$$b_\sigma(k, p) = \frac{\sqrt{2}}{s_{-\sigma}(k, \beta)} \left( s_{-\sigma}(\beta, \hat{p}) s_\sigma(\hat{p}, k) + \frac{m^2}{2\zeta \hat{p}} \sqrt{(2\beta\zeta)(2\zeta k)} \right)$$

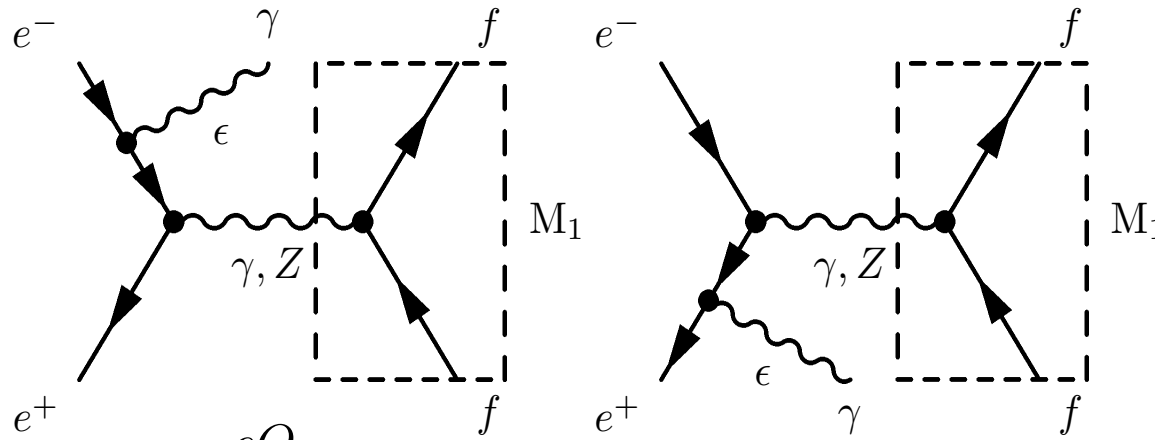
Thanks to the above diagonality we easily obtain/explore the soft limit of the multi-photon amplitudes.



# CEEX Amplitudes

III-6

## ISR First order real 1-photon



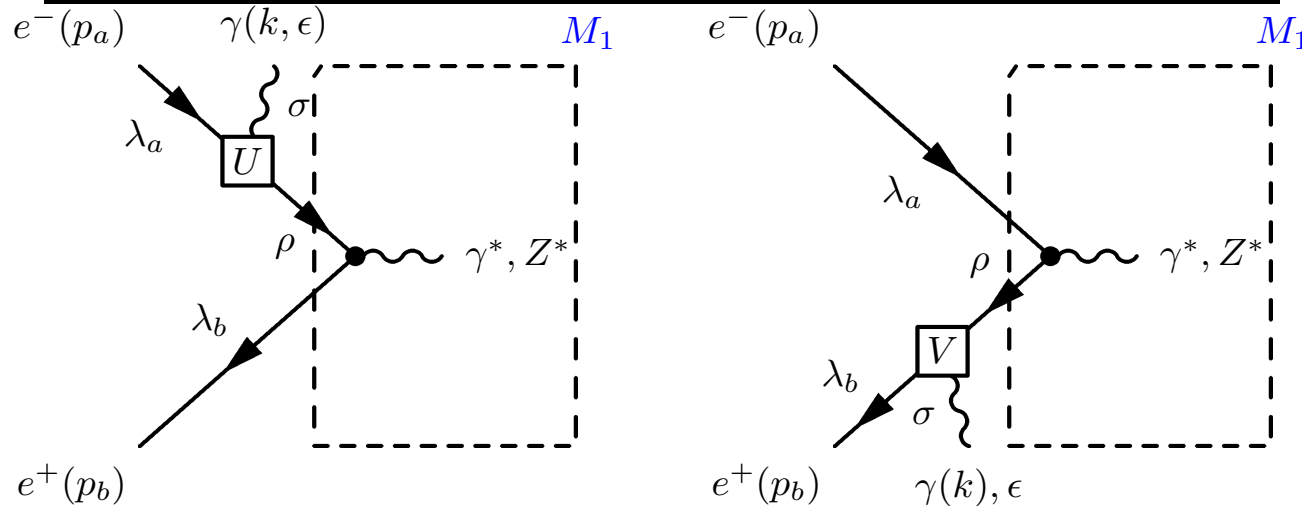
$$\mathcal{M}_1^{\text{ISR}} \left( \begin{matrix} p_a & p_b & k_1 \\ \lambda_a & \lambda_b & \sigma_1 \end{matrix} \right) = \frac{eQ_e}{2k_1 p_a} \bar{v}(p_b, \lambda_b) \mathbf{M}_1 (\not{p}_a + m - \not{k}_1) \not{\epsilon}_{\sigma_1}^*(k_1) u(p_a, \lambda_a) \\ + \frac{eQ_e}{2k_1 p_b} \bar{v}(p_b, \lambda_b) \not{\epsilon}_{\sigma_1}^*(k_1) (-\not{p}_b + m + \not{k}_1) \mathbf{M}_1 u(p_a, \lambda_a)$$

- Separate parts containing  $\not{p} \pm m$  and  $\not{k}_1$ , (IR and finite)
- Apply completeness relation for  $\not{p} \pm m$  and  $\not{k}_1$ ,
- Express bispinor components in terms of  $U$  and  $V$  matrices.
- Use diagonality property.

# CEEX Amplitudes

III-7

## ISR in terms of U and V matrices; separation of IR



Using compact notation  $\begin{bmatrix} p \\ \lambda \end{bmatrix} \equiv \begin{bmatrix} p_b & p_a & p_c & p_d \\ \lambda_b & \lambda_a & \lambda_c & \lambda_d \end{bmatrix} \equiv \begin{bmatrix} p_b & p_a \\ \lambda_b & \lambda_a \end{bmatrix}_{[cd]}$  we get:

$$\begin{aligned} \mathcal{M}_1^{\text{ISR}} \begin{pmatrix} p_a & p_b & p_c & p_d & k_1 \\ \lambda_a & \lambda_b & \lambda_c & \lambda_d & \sigma_1 \end{pmatrix} &\equiv \mathcal{M}_1^{\text{ISR}} \begin{pmatrix} p & k_1 \\ \lambda & \sigma_1 \end{pmatrix} = \\ &= \frac{eQ_e}{2k_1 p_a} \sum_{\rho} \mathfrak{B} \begin{bmatrix} p_b & p_a \\ \lambda_b & \rho \end{bmatrix}_{[cd]} U \begin{bmatrix} p_a & k_1 & p_a \\ \rho & \sigma_1 & \lambda_a \end{bmatrix} - \frac{eQ_e}{2k_1 p_b} \sum_{\rho} V \begin{bmatrix} p_b & k_1 & p_b \\ \lambda_b & \sigma_1 & \rho \end{bmatrix} \mathfrak{B} \begin{bmatrix} p_b & p_a \\ \rho & \lambda_a \end{bmatrix}_{[cd]} \\ &- \frac{eQ_e}{2k_1 p_a} \sum_{\rho} \mathfrak{B} \begin{bmatrix} p_b & k_1 \\ \lambda_b & \rho \end{bmatrix}_{[cd]} U \begin{bmatrix} k_1 & k_1 & p_a \\ \rho & \sigma_1 & \lambda_a \end{bmatrix} + \frac{eQ_e}{2k_1 p_b} \sum_{\rho} V \begin{bmatrix} p_b & k_1 & k_1 \\ \lambda_b & \sigma_1 & \rho \end{bmatrix} \mathfrak{B} \begin{bmatrix} k_1 & p_a \\ \rho & \lambda_a \end{bmatrix}_{[cd]} \end{aligned}$$

Now we may exploit **diagonality** of U and V in **IR parts**, see next slide  $\rightarrow$

## Real 1- $\gamma$ ISR summary

IR and non-IR clearly separated:

$$\mathcal{M}\left(\begin{matrix} p & k_1 \\ \lambda & \sigma_1 \end{matrix}\right) = \begin{array}{c} \begin{array}{c} a \\ \swarrow \\ \bullet \\ \downarrow \\ \bullet \\ \swarrow \\ b \end{array} \begin{array}{c} 1 \\ \swarrow \\ \bullet \\ \downarrow \\ \bullet \\ \swarrow \\ c \end{array} \\ \begin{array}{c} a \\ \swarrow \\ \bullet \\ \downarrow \\ \bullet \\ \swarrow \\ b \end{array} \begin{array}{c} \gamma^* z^* \\ \swarrow \\ \bullet \\ \downarrow \\ \bullet \\ \swarrow \\ d \end{array} \end{array} = \mathbf{s}_{[1]}^{\{1\}} \mathfrak{B}\left[\begin{matrix} p \\ \lambda \end{matrix}\right] + r^{\{1\}}\left(\begin{matrix} p & k_1 \\ \lambda & \sigma_1 \end{matrix}\right)$$

$$\begin{aligned} \mathbf{s}_{[1]}^{\{1\}} &\equiv \mathbf{s}_{\sigma_1}^{\{1\}}(k_1) \equiv \mathbf{s}_{[1]}^{(a)} + \mathbf{s}_{[1]}^{(b)}, \\ \mathbf{s}_{[1]}^{(a)} &= eQ_e \frac{b_\sigma(k, p_a)}{-2k_1 p_a}, \quad \mathbf{s}_{[1]}^{(b)} = eQ_e \frac{b_\sigma(k, p_b)}{2k_1 p_b}, \\ r^{\{1\}}\left(\begin{matrix} p & k_1 \\ \lambda & \sigma_1 \end{matrix}\right) &= -\frac{eQ_e}{2k_1 p_a} \sum_{\sigma'_1} \mathfrak{B}\left[\begin{matrix} p_b & k_1 \\ \lambda_b & \sigma'_1 \end{matrix}\right]_{[cd]} U\left[\begin{matrix} k_1 & k_1 & p_a \\ \sigma'_1 & \sigma_1 & \lambda_a \end{matrix}\right] \\ &\quad + \frac{eQ_e}{2k_1 p_b} \sum_{\sigma'_1} V\left[\begin{matrix} p_b & k_1 & k_1 \\ \lambda_b & \sigma_1 & \sigma'_1 \end{matrix}\right] \mathfrak{B}\left[\begin{matrix} k & p_a \\ \sigma'_1 & \lambda_a \end{matrix}\right]_{[cd]} \end{aligned}$$

Introduce super-compact notation:

$$(if) = 2k_i \cdot p_f, \quad f = a, b, c, d,$$

$$[p] \equiv \left[\begin{matrix} p_b & p_a & p_c & p_d \\ \lambda_b & \lambda_a & \lambda_c & \lambda_d \end{matrix}\right] \equiv \left[\begin{matrix} p_b & p_a \\ \lambda_b & \lambda_a \end{matrix}\right]_{[cd]} \equiv [ba] \left[\begin{matrix} p_c & p_d \\ \lambda_c & \lambda_d \end{matrix}\right] \equiv [bacd]$$

and summation understood over **primed** (spin) indices.

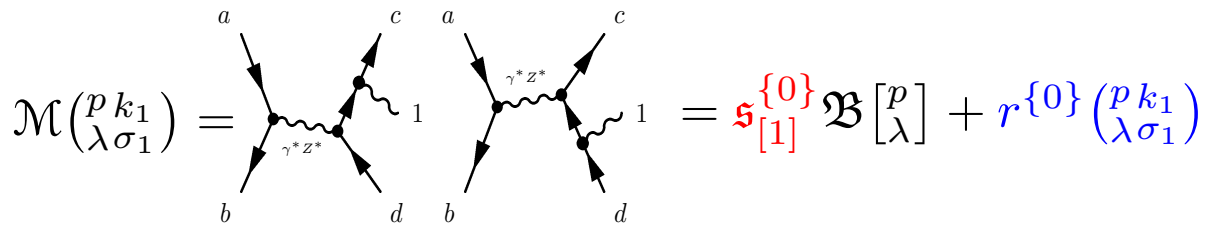
An get super-compact expression:

$$r^{\{1\}}\left(\begin{matrix} p & k_1 \\ \lambda & \sigma_1 \end{matrix}\right) = \frac{eQ_e}{-(1a)} \mathfrak{B}_{[b1']_{[cd]}} U_{[1'1a]} + \frac{eQ_e}{(1b)} V_{[b11']} \mathfrak{B}_{[1'a]_{[cd]}}$$

This notation will be useful for 2 and more photons.

## Real 1- $\gamma$ FSR summary

IR and non-IR clearly separated:



$$\mathcal{M}\left(\begin{matrix} p & k_1 \\ \lambda & \sigma_1 \end{matrix}\right) = \text{[Diagrams]} = \mathbf{s}_{[1]}^{\{0\}} \mathfrak{B}\left[\begin{matrix} p \\ \lambda \end{matrix}\right] + r^{\{0\}} \left(\begin{matrix} p & k_1 \\ \lambda & \sigma_1 \end{matrix}\right)$$

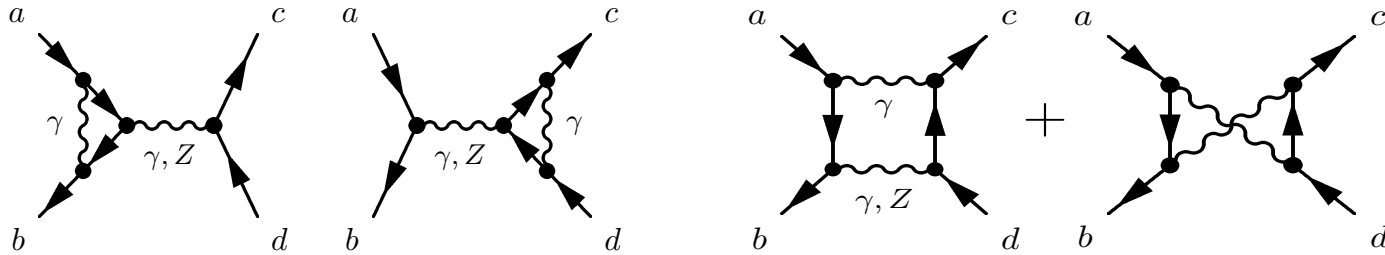
Using super-compact notation:

$$\mathbf{s}_{[1]}^{\{0\}} \equiv \mathbf{s}_{\sigma_1}^{\{0\}}(k_1) \equiv \mathbf{s}_{[1]}^{(c)} + \mathbf{s}_{[1]}^{(d)},$$

$$\mathbf{s}_{[1]}^{(c)} = eQ_f \frac{b_\sigma(k, p_c)}{2k_1 p_c}, \quad \mathbf{s}_{[1]}^{(d)} = eQ_f \frac{b_\sigma(k, p_d)}{-2k_1 p_d},$$

$$r^{\{0\}} \left(\begin{matrix} p & k_1 \\ \lambda & \sigma_1 \end{matrix}\right) = \frac{eQ_f}{(1c)} U_{[c11']} \mathfrak{B}_{[ba][1'd]} + \frac{eQ_f}{-(1d)} \mathfrak{B}_{[ba][c1']} V_{[1'1d]}$$

## First order, one virtual photon



$$\mathcal{M}_0^{(1)}(p_\lambda; X) = \mathfrak{B}(p_\lambda; X) [1 + Q_e^2 F_1(s, m_\gamma) + Q_f^2 F_1(s, m_\gamma)] + \mathcal{M}_{\text{box}}(p_\lambda; X),$$

$F_1$  is the electric formfactor regularized with  $m_\gamma$ , keep exact final fermion mass.

We omit  $F_2$ , this is justified for light final fermions. To be restored later on.

Spin amplitudes for  $\gamma$ - $\gamma$  and  $\gamma$ -Z boxes are:

$$\begin{aligned} \mathcal{M}_{\text{Box}}(p_\lambda; X) = & \\ = ie^2 \sum_{B=\gamma, Z} & \frac{g_{\lambda_a}^{e,B} g_{-\lambda_a}^{f,B} T_{\lambda_c \lambda_a} T'_{\lambda_b \lambda_d} + g_{\lambda_a}^{e,B} g_{\lambda_a}^{f,B} U'_{\lambda_c \lambda_b} U_{\lambda_a \lambda_d}}{X^2 - M_B^2 + i\Gamma_B X^2 / M_B} \delta_{\lambda_a, -\lambda_b} \delta_{\lambda_c, -\lambda_d} \\ \times \frac{\alpha}{\pi} Q_e Q_f & [\delta_{\lambda_a, \lambda_c} f_{\text{BDP}}(\bar{M}_B^2, m_\gamma, s, t, u) - \delta_{\lambda_a, -\lambda_c} f_{\text{BDP}}(\bar{M}_B^2, m_\gamma, s, u, t)], \end{aligned}$$

where  $\bar{M}_Z^2 = M_Z^2 - iM_Z \Gamma_Z$ ,  $\bar{M}_\gamma^2 = m_\gamma^2$ . and function  $f_{\text{BDP}}$  is defined in Brown, Decker and Paschos (1984); Mandelstam variables  $s$ ,  $t$  and  $u$  defined as usual.

Warm up example: resonant CEEX, zero-th order  $\mathcal{O}(\alpha^0)_{\text{CEEX}}$

$$\sigma^{(0)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_{n+2}(p_c, p_d, k_1, \dots, k_n) \frac{1}{4} \sum_{\lambda, \sigma_j} |\mathcal{M}_n^{(r)}(p \begin{smallmatrix} k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{smallmatrix})|^2$$

$$\mathcal{M}_n^{(0)}(p \begin{smallmatrix} k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{smallmatrix}) = e^{\alpha B_4^*(p_a, \dots, p_d)} \sum_{\wp \in \mathcal{P}} \frac{X_{\wp}^2}{s_{cd}} \mathfrak{B}(p; X_{\wp}) \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\wp_i\}}$$

Notation:  $\binom{p}{\lambda} \equiv \binom{p_a \ p_b \ p_c \ p_d}{\lambda_a \ \lambda_b \ \lambda_c \ \lambda_d}$ ,  $s_{cd} = (p_c + p_d)^2$ ,  $d\tau_n \equiv$  Lorentz inv. phase space.

- The *coherent sum* is taken over set of  $2^n$  partitions:  $\mathcal{P} = \{ (0, 0, 0, \dots, 0), (1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, 1, \dots, 1) \}$ .
- In single partition  $\wp = (\wp_1, \dots, \wp_i, \dots, \wp_n)$ ,  $\wp_i = 1, 0$  for ISR, FSR so that  $X_{\wp} = p_a + p_b - \sum_{i=1}^n \wp_i k_i \equiv$  the 4-mom. in **resonance** propagator.
- $\mathfrak{s}_{[i]}^{\{\omega\}} \equiv j_{\{\omega\}}(k_i) \cdot \epsilon_{\sigma_i}$  are **IR**-div. real-photon soft-factors with ISR/FSR currents:  $j_{\{1\}}^{\mu}(k_i) = -\frac{eQ_e}{2} \left( \frac{p_a^{\mu}}{k_i p_a} - \frac{p_b^{\mu}}{k_i p_b} \right)$ ,  $j_{\{0\}}^{\mu}(k_i) = \frac{eQ_e}{2} \left( \frac{p_c^{\mu}}{k_i p_c} - \frac{p_d^{\mu}}{k_i p_d} \right)$ , and  $\epsilon_{\sigma_i}$  are our KS/Beijing photon polarization vectors (axial gauge).
- $B_4^*(p_a, \dots, p_d)$  is **IR** virtual form-factor with the **resonance** part, see next slide.

## Virtual Formfactor

**Factorization of virtual IR by Yennie-Frautschi-Suura (1961):**

$$\sum_{n=0}^{\infty} \text{Diagram}_n = e^{\alpha B_4} \text{Diagram}_{\gamma, Z} \times (1 + \Delta_{\text{fin.}})$$

where  $B_4(p_a, \dots, p_d) = \int \frac{d^4 k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} |J_I(k) - J_F(k)|^2,$

$$J_I = eQ_e (\hat{J}_a(k) - \hat{J}_b(k)), \quad J_F = eQ_f (\hat{J}_c(k) - \hat{J}_d(k)), \quad \hat{J}_f^\mu(k) \equiv \frac{2p_f^\mu + k^\mu}{k^2 + 2kp_f + i\epsilon}$$

**Modification for Z resonance by Greco, Pancheri and Srivastava (1975,1980):**

$B_4 \rightarrow B_4^*$

[Notation:  $M^2 = M_Z^2 - iM\Gamma$ ]

$$|J_I - J_F|^2 \rightarrow |J_I(k)|^2 + |J_F(k)|^2 - 2\Re(J_I(k) \cdot J_F^*(k)) \frac{(p_a + p_b)^2 + M^2}{(p_a + p_b - k)^2 + M^2}.$$

Only diagrams with Z propagator are concerned:

$$e^{\alpha B_4^*} \text{Diagram}_{Z} \times (1 + \Delta_{\text{fin.}}) + e^{\alpha B_4} \text{Diagram}_{\gamma} \times (1 + \Delta'_{\text{fin.}})$$

## Limit (1): no resonances at all

$$\sum_{\wp \in \mathcal{P}} e^{\alpha B_4^*(X_\wp)} \frac{X_\wp^2}{s_{cd}} \mathfrak{B}(p_\lambda; X_\wp) \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\wp_i\}} \implies e^{\alpha B_4^*} \mathfrak{B}(p_\lambda; P) \prod_{i=1}^n (\mathfrak{s}_{[i]}^{\{0\}} + \mathfrak{s}_{[i]}^{\{1\}}),$$

because  $\sum_{\wp \in \mathcal{P}} \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\wp_i\}} \equiv \prod_{i=1}^n (\mathfrak{s}_{[i]}^{\{0\}} + \mathfrak{s}_{[i]}^{\{1\}})$ . [ $P = p_a + p_b$ , for example.]

Nevertheless it is good to keep sum over partition even for the non-resonant case! (Better LL summations).

CEEX for non-resonant is also very valuable! (The only hope for Bhabha!)

## Limit (2): very narrow resonance

$$|\mathcal{M}_n^{(0)}|^2 = \sum_{\wp \in \mathcal{P}} \sum_{\wp' \in \mathcal{P}} e^{\alpha B_4^*(X_\wp)} e^{\alpha (B_4^*(X_{\wp'}))^*} \mathfrak{B}(p_\lambda; X_\wp) \mathfrak{B}(p_\lambda; X_{\wp'})^* \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\wp_i\}} \prod_{j=1}^n \mathfrak{s}_{[j]}^{\{\wp'_j\}}^*$$

$$\implies e^{2\alpha \Re B_2(p_a, p_b)} e^{2\alpha \Re B_2(p_c, p_d)} \sum_{\wp \in \mathcal{P}} |\mathfrak{B}(p_\lambda; X_\wp)|^2 \prod_{i=1}^n |\mathfrak{s}_{[i]}^{\{\wp_i\}}|^2.$$

Neglect of ISR\*FSR interferences implies that terms  $\wp \neq \wp'$  drop out.

This approximation as used in KORALZ/YFS3 for LEP1.

At LEP2 it cannot be justified any more.



## Full Scale CEEX $\mathcal{O}(\alpha^r)$ master formula

### Polarized total x-section:

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_n(p_a + p_b; p_c, p_d, k_1, \dots, k_n) e^{2\alpha\Re B_4} \sum_{\sigma_i, \lambda, \bar{\lambda}} \sum_{i,j,l,m=0}^3$$

$$\hat{\varepsilon}_a^i \hat{\varepsilon}_b^j \sigma_{\lambda_a \bar{\lambda}_a}^i \sigma_{\lambda_b \bar{\lambda}_b}^j \mathfrak{M}_n^{(r)} \left( \begin{matrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{matrix} \right) \left[ \mathfrak{M}_n^{(r)} \left( \begin{matrix} p & k_1 & k_2 & \dots & k_n \\ \bar{\lambda} & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{matrix} \right) \right]^* \sigma_{\bar{\lambda}_c \lambda_c}^l \sigma_{\bar{\lambda}_d \lambda_d}^m \hat{h}_c^l \hat{h}_c^m$$

### CEEX amplitudes:

$$\mathfrak{M}_n^{(1)} \left( \begin{matrix} p & k_1 & \dots & k_n \\ \lambda & \sigma_1 & \dots & \sigma_n \end{matrix} \right) = \sum_{\varphi \in \mathcal{P}} \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\varphi_i\}} \left\{ \beta_0^{(1)} \left( \begin{matrix} p \\ \lambda \end{matrix}; X_\varphi \right) + \sum_{j=1}^n \frac{\beta_{1\{\varphi_j\}}^{(1)} \left( \begin{matrix} p & k_j \\ \lambda & \sigma_j \end{matrix}; X_\varphi \right)}{\mathfrak{s}_{[j]}^{\{\varphi_j\}}} \right\}$$

$$\mathfrak{M}_n^{(2)} \left( \begin{matrix} p & k_1 & \dots & k_n \\ \lambda & \sigma_1 & \dots & \sigma_n \end{matrix} \right) = \sum_{\varphi \in \mathcal{P}} \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\varphi_i\}} \times \left\{ \beta_0^{(2)} \left( \begin{matrix} p \\ \lambda \end{matrix}; X_\varphi \right) + \sum_{j=1}^n \frac{\beta_{2\{\varphi_j\}}^{(1)} \left( \begin{matrix} p & k_j \\ \lambda & \sigma_j \end{matrix}; X_\varphi \right)}{\mathfrak{s}_{[j]}^{\{\varphi_j\}}} + \sum_{1 \leq j < l \leq n} \frac{\beta_{2\{\varphi_j, \varphi_l\}}^{(2)} \left( \begin{matrix} p & k_j & k_l \\ \lambda & \sigma_j & \sigma_l \end{matrix}; X_\varphi \right)}{\mathfrak{s}_{[j]}^{\{\varphi_j\}} \mathfrak{s}_{[l]}^{\{\varphi_l\}}} \right\}$$

For details see next slides.

## SPIN STRUCTURE

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_n(p_a + p_b; p_c, p_d, k_1, \dots, k_n) e^{2\Re B_4} \sum_{\sigma_i, \lambda, \bar{\lambda}} \sum_{i, j, l, m=0}^3$$

$$\hat{\varepsilon}_a^i \hat{\varepsilon}_b^j \sigma_{\lambda_a \bar{\lambda}_a}^i \sigma_{\lambda_b \bar{\lambda}_b}^j \mathfrak{M}_n^{(r)} \left( \begin{matrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{matrix} \right) \left[ \mathfrak{M}_n^{(r)} \left( \begin{matrix} p & k_1 & k_2 & \dots & k_n \\ \bar{\lambda} & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{matrix} \right) \right]^* \sigma_{\bar{\lambda}_c \lambda_c}^l \sigma_{\bar{\lambda}_d \lambda_d}^m \hat{h}_c^l \hat{h}_c^m$$

- $\sigma^k$  for  $k = 1, 2, 3$  are Pauli matrices and  $\sigma_{\lambda, \mu}^0 = \delta_{\lambda, \mu}$  is unit matrix.
- $\hat{\varepsilon}_1^a, \hat{\varepsilon}_2^b, a, b = 1, 2, 3$  are the components of the conventional spin polarization vectors of the **beam**  $e^\pm$  and  $\hat{\varepsilon}_i^0 \equiv 1$  in  $e^\pm$  rest frame [ $\hat{\varepsilon}_i \cdot p_i = m_e$ ].
- $\hat{h}_3^c \hat{h}_4^d$  are polarimeter vectors of outgoing fermions. They carry spin information to **final fermion decay** processes [Notation:  $\hat{h}_i \cdot p_i = m_f$ ].
- $\hat{\varepsilon}_i^a$  and  $\hat{h}_i^a$  are **primarily** defined in the so called GPS frames of the corresponding fermions. (Important for the use of Pauli matrices!)

## $\mathcal{O}(\alpha^1)_{\text{CEEX}}$ Beta's

$$\mathfrak{M}_n^{(1)}(p, k_1, \dots, k_n; \lambda, \sigma_1, \dots, \sigma_n) = \sum_{\varphi \in \mathcal{P}} \prod_{i=1}^n \mathfrak{s}_{[i]}^{\{\varphi_i\}} \left\{ \beta_0^{(1)}(p; X_\varphi) + \sum_{j=1}^n \frac{\beta_{1\{\varphi_j\}}^{(1)}(p, k_j; X_\varphi)}{\mathfrak{s}_{[j]}^{\{\varphi_j\}}} \right\}$$

$$\beta_0^{(1)}(p; X) = \mathfrak{B}(p; X) (1 + \delta_{\text{Virt}}^{(1)}) + \mathcal{R}_{\text{Box}}(p; X)$$

$$\beta_{1\{1\}}^{(1)}(p, k_j; X) = r^{\{1\}}(p, k_j; X)$$

$$\beta_{1\{0\}}^{(1)}(p, k_j; X) = r^{\{0\}}(p, k_j; X) + \left( \frac{(p_c + p_d + k_j)^2}{(p_c + p_d)^2} - 1 \right) \mathfrak{B}(p; X)$$

**The most important:  $\beta$ 's are IR-finite!**

Formal definition (omitting inessential arguments/indices) of  $\beta$ 's are :

$$\beta_0^{(1)} = (e^{\alpha B_4} \mathcal{M}_0^{(1)})|_{\mathcal{O}(\alpha^1)}$$

$$\beta_{1\{\omega\}}^{(1)}(k_j; \sigma_j) = \mathcal{M}_{1\{\omega\}}^{(1)}(k_j; \sigma_j) - \beta_0^{(1)} \mathfrak{s}_{[j]}^{\{\omega\}}$$

Complications due to **Z resonance** see next slide.

## Complication due to resonance

In amplitude part **proportional** to resonant propagator  we replace:

$$B_4 \Rightarrow B_4^* = B_I(p_a, p_b) + B_F(p_c, p_b) + B_{IFI}^*(p_a, p_b, p_c, p_b, X_\varphi)$$

$$\int d^4k |J_I - J_F|^2 \Rightarrow \int d^4k |J_I(k)|^2 + |J_F(k)|^2 - 2\Re(J_I(k) \cdot J_F^*(k)) \frac{X_\varphi^2 + M^2}{(X_\varphi - k)^2 + M^2}.$$

[Notation:  $M^2 = M_Z^2 - iM\Gamma$ ]

**Consequently:**

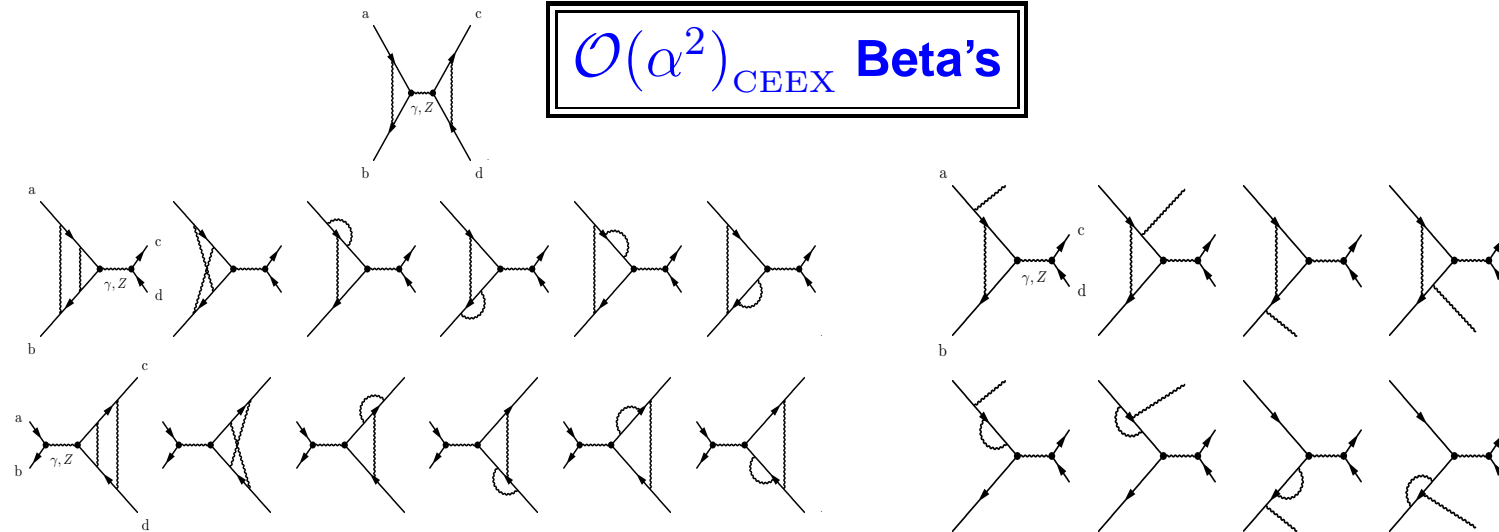
$$\beta_0^{(1)} = \left( e^{2\alpha B_4^*} \mathcal{M}_0^{(1)} \right) \Big|_{\mathcal{O}(\alpha^1)} \text{ induces additional subtraction of order } \ln \frac{\Gamma_Z}{M_Z} \text{ in } \mathcal{R}_{\text{Box}}.$$

The overall virtual formfactor in the master formula gains partition dependence in the

resonance part:  $e^{2\alpha B_4^*} = e^{2\alpha B_4(p_a, p_b, p_c, p_d)} + 2\alpha \delta B_4^*(p_a, p_b, p_c, p_d; X_\varphi).$

# CEEX Amplitudes

III-18



$$\beta_0^{(l)}(p; P) = \mathfrak{M}_0^{(l)}(p; P), \quad l = 0, 1, 2,$$

$$\beta_{1\{I\}}^{(1+l)}(p, k_1; P - k_1) = \mathfrak{M}_{1\{I\}}^{(1+l)}(p, k_1; P - k_1) - \beta_0^{(l)}(p; P - k_1) \mathfrak{s}_{\sigma_1}^{\{I\}}(k_1), \quad l = 0, 1,$$

$$\beta_{1\{F\}}^{(1+l)}(p, k_1; P) = \mathfrak{M}_{1\{F\}}^{(1+l)}(p, k_1; P) - \beta_0^{(l)}(p; P) \mathfrak{s}_{\sigma_1}^{\{F\}}(k_1), \quad l = 0, 1,$$

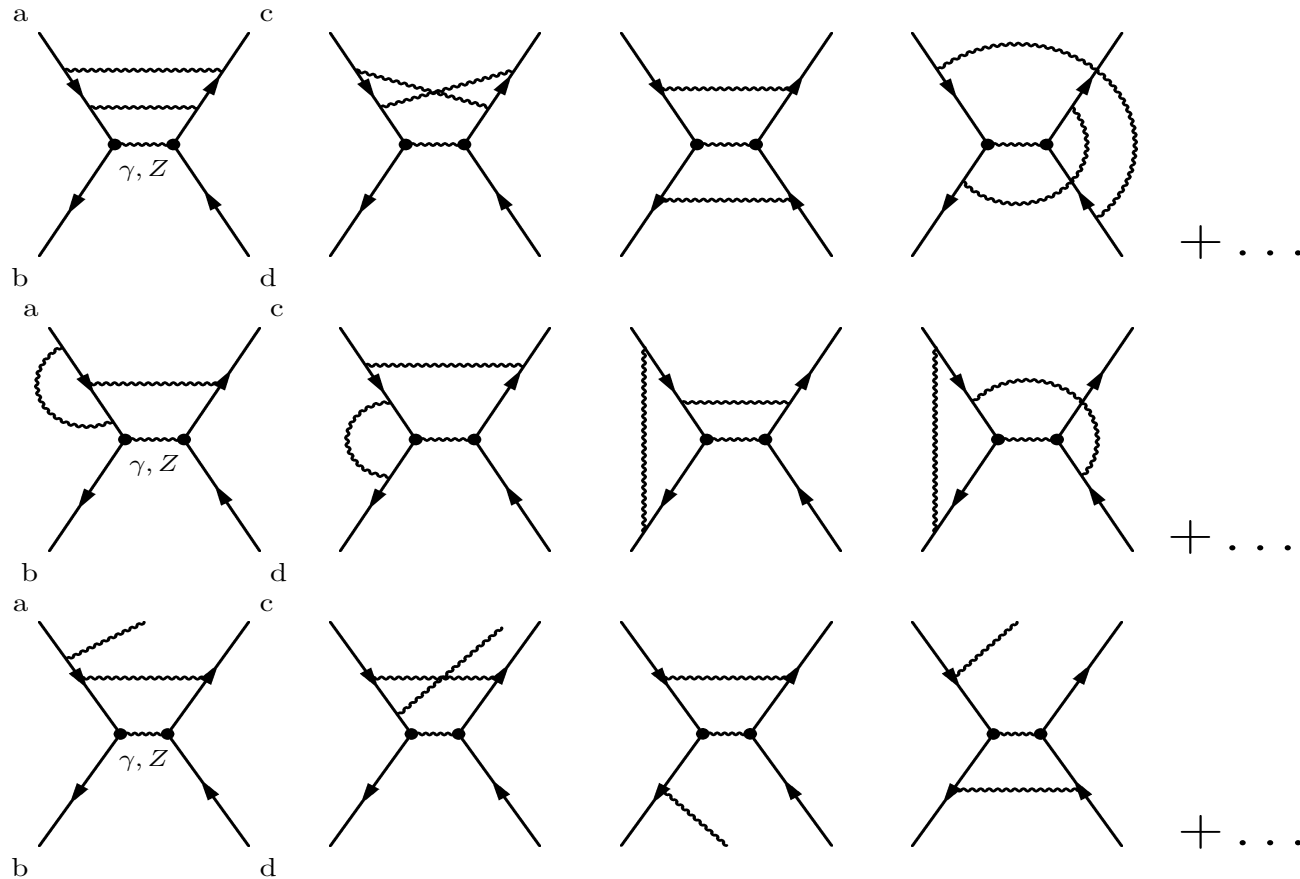
$$\beta_{2\{\omega_1, \omega_2\}}^{(2)}(p, k_1, k_2; X_\omega) = \mathfrak{M}_{2\{\omega_1, \omega_2\}}^{(2)}(p, k_1, k_2; X_\omega)$$

$$- \beta_{1\{\omega_1\}}^{(1)}(p, k_1; X_\omega) \mathfrak{s}_{\sigma_2}^{\{\omega_2\}}(k_2) - \beta_{1\{\omega_2\}}^{(1)}(p, k_2; X_\omega) \mathfrak{s}_{\sigma_1}^{\{\omega_1\}}(k_1) - \beta_0^{(0)}(p; X_\omega) \mathfrak{s}_{\sigma_1}^{\{\omega_1\}}(k_1) \mathfrak{s}_{\sigma_2}^{\{\omega_2\}}(k_2) \quad (1)$$

# CEEX Amplitudes

III-19

$\mathcal{O}(\alpha^2)_{\text{CEEX}}$  Shopping list



**Never ever combine with real emission! (Keep photon mass and publish.)**

**Never ever square and spin sum! In the M.C. real emission is added anyway.**

## Traditional real photon regulator.

Optionally, the traditional IR-cut  $k^0 > \varepsilon\sqrt{s}/2$  on real  $\gamma$ 's can be introduced.

Phase space integral  $m_\gamma < k^0 < \varepsilon\frac{\sqrt{s}}{2}$  done analitically (rigorously)  $\Rightarrow e^{2\alpha\tilde{B}_4}$  factor,

where:  $\tilde{B}_4(p_a, \dots, p_d) = Q_e^2 \tilde{B}_2(p_a, p_b) + Q_f^2 \tilde{B}_2(p_c, p_d)$

+  $Q_e Q_f \tilde{B}_2(p_a, p_c) + Q_e Q_f \tilde{B}_2(p_b, p_d) - Q_e Q_f \tilde{B}_2(p_a, p_d) - Q_e Q_f \tilde{B}_2(p_b, p_c)$ ,

$$\tilde{B}_2(p, q) \equiv \int_{k^0 < \varepsilon\sqrt{s}/2} \frac{d^3k}{k^0} \frac{(-1)}{8\pi^2} \left( \frac{p}{kp} - \frac{q}{kq} \right)^2.$$

## New Master Formula:

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{k_i^0 > \varepsilon\sqrt{s}/2} d\tau_n(p_a + p_b; p_c, p_d, k_1, \dots, k_n) e^{2\alpha\Re B_4 + 2\alpha\tilde{B}_4} \sum_{\sigma_i, \lambda, \bar{\lambda}} \sum_{i, j, l, m=0}^3$$

$$\hat{\varepsilon}_a^i \hat{\varepsilon}_b^j \sigma_{\lambda_a \bar{\lambda}_a}^i \sigma_{\lambda_b \bar{\lambda}_b}^j \mathfrak{M}_n^{(r)} \left( \begin{matrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{matrix} \right) \left[ \mathfrak{M}_n^{(r)} \left( \begin{matrix} p & k_1 & k_2 & \dots & k_n \\ \bar{\lambda} & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{matrix} \right) \right]^* \sigma_{\bar{\lambda}_c \lambda_c}^l \sigma_{\bar{\lambda}_d \lambda_d}^m \hat{h}_c^l \hat{h}_c^m$$

The resonant part (if present) is multiplied by  $e^{2\alpha\delta B_4^*(p_a, p_b, p_c, p_d; X_\phi)}$

## IR-cancellations

- IR-cancellations in case of traditional energy-cut regulator is manifest.
- In original master formula with  $m_\gamma$  regulator one may check IR-finiteness with by **analytical** partial differentiation with respect to the photon mass:  
 $\partial\sigma^{(r)}/\partial m_\gamma = 0$ . (G. Burgers (1989) unpublished).
- The classical method of YFS (1961) relies on the techniques of the Melin transform.

NB. Version with  $m_\gamma$  regulator is perfectly implementable in the M.C.



## CPU considerations: photon spin randomization

The single spin amplitude  $\mathfrak{M}_n^{(1)}$  contains already  $2^n(n+1)$  terms ( $2^n$  due to ISR/FSR partitions). The grand sum over spins counts  $2^n 4^4 4^4 = 2^{n+16}$  terms!!! Altogether we expect up to  $N \sim n2^{2n+16}$  operations in the CPU time expensive complex (16bytes) arithmetics. Typically in  $e^-e^+ \rightarrow \mu^-\mu^+$  the average photon multiplicity with  $k^0 > 1MeV$  is about 3, corresponding to  $N \sim 10^7$  terms. In a sample of  $10^4$  MC events there will be a couple events with  $n = 10$  and  $N = 10^{12}$  terms. Partial solutions:  $\sum_a \varepsilon_i^a \sigma_{\lambda\bar{\lambda}}^a$  and the  $\mathfrak{s}$ -factors evaluated only once, stored and reused (save  $2^8$ ).

The trick of **photon spin randomization** speeds up substantially the numerical calculation in the Monte Carlo program: Instead of evaluating the sum over photon spins  $\sigma_i, i = 1, \dots, n$  we generate randomly one spin sequence of  $(\sigma_1, \dots, \sigma_n)$  per MC event and the MC weight is calculated only for this particular spin sequence! We save one hefty  $2^n$  factor in the CPU time!

The formal proof of the correctness of this method can be found in Sect. 4 of “**Guide to practical Monte Carlo methods**”, (1998), <http://wwwcn.cern/~jadach>.

## APPENDIX A: More on $\mathcal{O}(\alpha^1)_{\text{CEEX}}$ virtual's

The IR-finite  $\delta_{\text{Virt}}^{(1)}$  reads explicitly:

$$\delta_{\text{Virt}}^{(1)}(s) = Q_e^2 F_1(s, m_\gamma) + Q_f^2 F_1(s, m_\gamma) - Q_e^2 \alpha B_2(s, m_\gamma) - Q_f^2 \alpha B_2(s, m_\gamma).$$

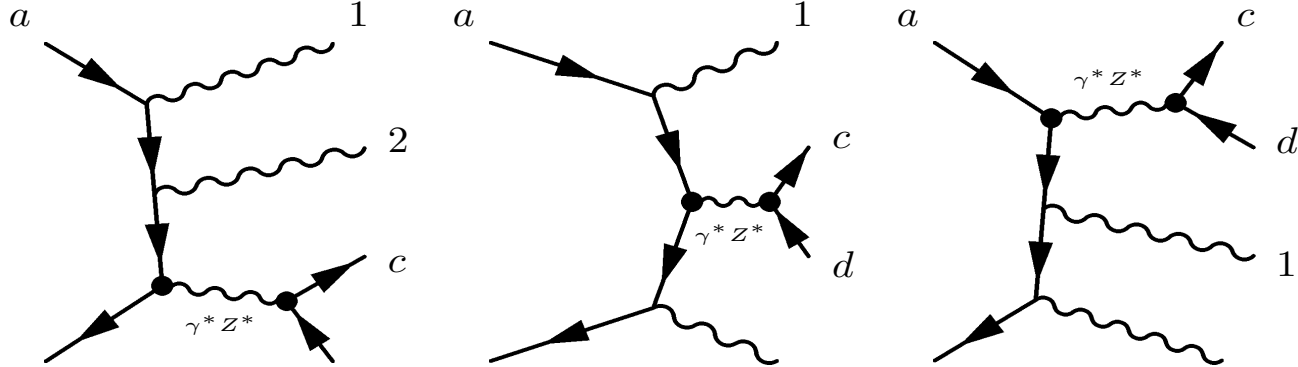
The  $\mathcal{R}_{\text{Box}}$  is effectively obtained from  $\mathcal{M}_{\text{Box}}$  by:

$$f_{\text{BDP}}(\bar{M}_B^2, m_\gamma, s, t, u) \Rightarrow f_{\text{BDP}}(\bar{M}_B^2, m_\gamma, s, t, u) - f_{\text{IR}}(m_\gamma, t, u), \text{ where}$$

$$f_{\text{IR}}(m_\gamma, t, u) = \frac{2}{\pi} B_2(m_\gamma, t) - \frac{2}{\pi} B_2(m_\gamma, u) = \ln\left(\frac{t}{u}\right) \ln\left(\frac{m_\gamma^2}{\sqrt{tu}}\right) + \frac{1}{2} \ln\left(\frac{t}{u}\right).$$

# Appendix B: Double bremsstr. ISR

III-24



$$\begin{aligned}
 \beta_{2\{11\}}^{(2)}(p, k_1, k_2) &= \mathfrak{M}_{2\{11\}}^{(2)}(p, k_1, k_2)^b - \beta_{1\{1\}}^{(1)}(p, k_1) \mathfrak{s}_{[2]}^{\{1\}} - \beta_{1\{1\}}^{(1)}(p, k_2) \mathfrak{s}_{[1]}^{\{1\}} - \beta_0^{(0)}(p) \mathfrak{s}_{[1]}^{\{1\}} \mathfrak{s}_{[2]}^{\{1\}} \\
 &= (eQ_e)^2 \left\{ \frac{\mathfrak{B}_{[ba']}[cd] U_{[a'12'']} - \mathfrak{B}_{[b1']}[cd] U_{[1'12'']} - \mathfrak{B}_{[b2']}[cd] U_{[2'12'']}}{- (1a) - (2a) + (12)} - \frac{U_{[2''2a]}}{- (2a)} \right. \\
 &\quad + \frac{V_{[b11'']}}{- (1b)} \frac{-V_{[1''2b']}}{- (1b) - (2b) + (12)} \mathfrak{B}_{[b'a][cd]} + \frac{V_{[1''21']}}{- (1b) - (2b) + (12)} \mathfrak{B}_{[1'a][cd]} + \frac{V_{[1''22']}}{- (1b) - (2b) + (12)} \mathfrak{B}_{[2'a][cd]} \\
 &\quad \left. + \frac{V_{[b11']}}{- (1b)} \mathfrak{B}_{[1'2']}[cd] \frac{-U_{[2'2a]}}{- (2a)} + (1 \leftrightarrow 2) \right\} \\
 &+ eQ_e \left\{ \frac{-\mathfrak{B}_{[b1']}[cd] U_{[1'1a]} - \mathfrak{B}_{[b2']}[cd] U_{[2'1a]}}{- (1a) - (2a) + (12)} \mathfrak{s}_{[2]}^{(a)} + \mathfrak{s}_{[1]}^{(b)} \frac{V_{[b22']}}{- (1a) - (2a) + (12)} \mathfrak{B}_{[2'a][cd]} + \frac{V_{[b21']}}{- (1a) - (2a) + (12)} \mathfrak{B}_{[1'a][cd]} \right. \\
 &\quad \left. - \mathfrak{s}_{[1]}^{(b)} \mathfrak{B}_{[b2']}[cd] \frac{U_{[2'2a]}}{- (2a)} + \frac{V_{[b11']}}{- (1b)} \mathfrak{B}_{[1'a][cd]} \mathfrak{s}_{[2]}^{(a)} + (1 \leftrightarrow 2) \right\} \\
 &+ \left( \mathfrak{s}_{[1]}^{(a)} \mathfrak{s}_{[2]}^{(a)} \frac{(12)}{(1a) + (2a) - (12)} + \mathfrak{s}_{[1]}^{(b)} \mathfrak{s}_{[2]}^{(b)} \frac{- (12)}{(1b) + (2b) + (12)} \right) \mathfrak{B}_{[\lambda]}^p.
 \end{aligned}$$

NOTATION :  $\mathfrak{s}_{[i]}^{(a)} = -eQ_e \frac{b_{\sigma_1}(k_i, p_a)}{2k_i p_a}$ ,  $\mathfrak{s}_{[i]}^{(b)} = +eQ_e \frac{b_{\sigma_1}(k_i, p_b)}{2k_i p_b}$ ,  $\mathfrak{s}_{[i]}^{\{1\}} \equiv \mathfrak{s}_{[i]}^{(a)} + \mathfrak{s}_{[i]}^{(b)}$   
 $(ia) \equiv k_i \cdot p_a$ ,  $(ib) \equiv k_i \cdot p_b$ . Implicit sums over primed indices!

## Conclusions for Part III

- First and second order CEEX is explicitly formulated
- For resonance the coherent treatment of ISR and FSR
- Exact matrix elements up to 2 photons (spinor techniques)
- Full treatment of spin, for beams and decaying final fermions

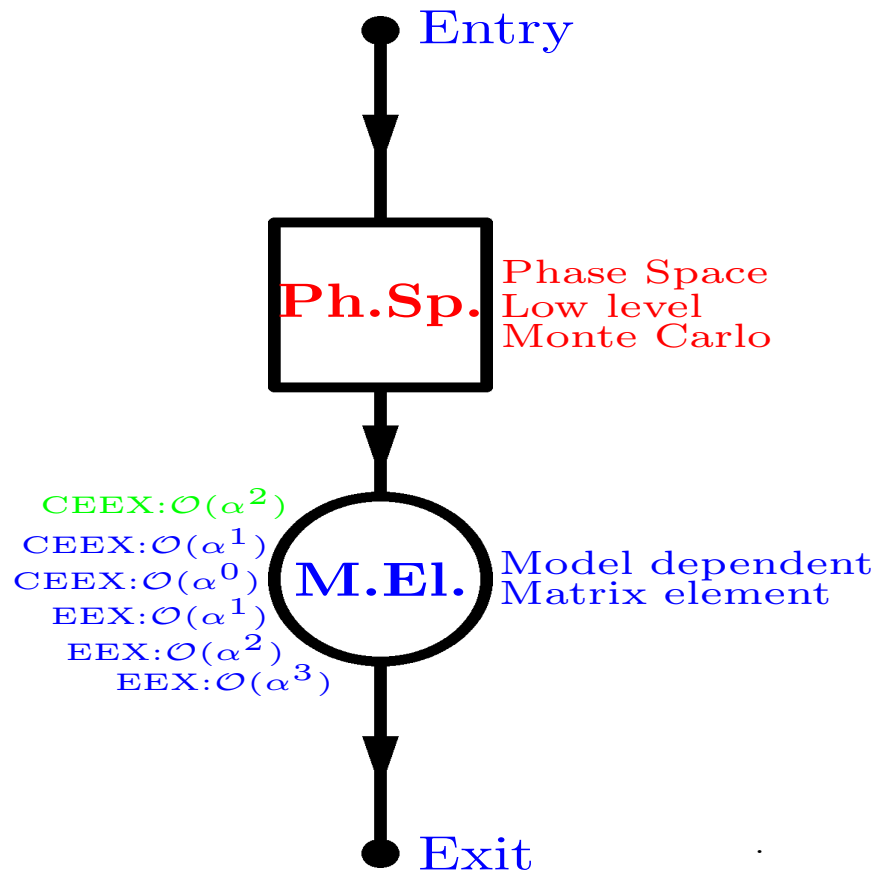
## Outline of Part IV

- General structure of the MC and its precision
  1. Basic test of technical precision
  2. Basic test of physical precision
- $\text{ISR} \otimes \text{FSR}$  interf. in  $\sigma_{tot}$  and  $A_{FB}$ 
  1. How big is  $\text{ISR} \otimes \text{FSR}$  interference in  $\sigma_{tot}, A_{FB}$ ?
  2. Do we know  $\text{ISR} \otimes \text{FSR}$  at  $\mathcal{O}(\alpha^1)$ ?
  3. Do we know  $\text{ISR} \otimes \text{FSR}$  beyond  $\mathcal{O}(\alpha^1)$ ?
  4. How sensitive  $\text{ISR} \otimes \text{FSR}$  is to cut-off changes?
  5. Answers from  $\mathcal{K}\mathcal{K}$  MC.
- Absolute predictions for  $\sigma_{tot}$   $A_{FB}$ 
  1. Comparisons with KORALZ,  $\mathcal{K}\mathcal{K}$ sem and Zfitter
  2. More tests of precision
- Summary

**THE PROBLEM: SM predictions for  $\sigma_{tot}$  and  $A_{FB}$  for  $e^-e^+ \rightarrow f\bar{f}$  are needed at the end of LEP2 with the precision 0.2 – 0.5%. CAN WE DELIVER?**

**For example the  $\text{ISR} \otimes \text{FSR}$  interference is already  $\sim 2\%$ .**

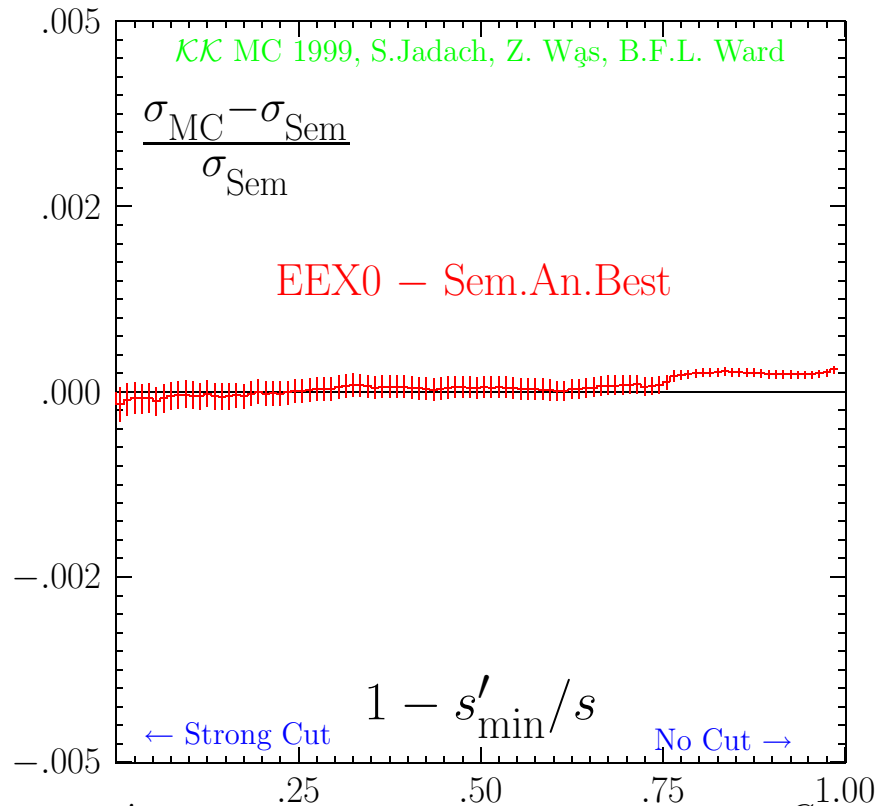
## General structure of $\mathcal{K}\mathcal{K}$ MC



- The program for  $e^-e^+ \rightarrow f\bar{f}$  is divided into two distinct parts/levels
- The tau decays and hadronization come after

## Baseline test of the Low level MC

**| Matrix Element |<sup>2</sup> = Born × Product of soft factors**



$$\sigma_{\text{Sem.}} = \int_0^{1-s'_{\text{min}}/s} dv \sigma_{\text{Born}}^f(s(1-u)(1-v)) \frac{e^{-C\gamma_e}}{\Gamma(1+\gamma_e)}$$

$$e^{\frac{1}{4}\gamma_e + \frac{\alpha}{\pi}(\frac{1}{2} + \frac{\pi^2}{3})} \gamma_e v^{\gamma_e-1} \left[ 1 - \frac{\gamma_e \ln(1-v)}{4} - \frac{\alpha \ln^2(1-v)}{\pi \cdot 2} + 0 \gamma_e^2 \right]$$

$$\frac{e^{-C\gamma_f}}{\Gamma(1+\gamma_f)} e^{\frac{1}{4}\gamma_f - \frac{1}{2}\gamma_f \ln(1-u) + \frac{\alpha}{\pi}(\frac{1}{2} + \frac{\pi^2}{3})} \gamma_f u^{\gamma_f-1} \left[ 1 - \frac{\gamma_f \ln(1-u)}{4} \right]$$

**Techn.Error = |MC - Sem.Anl.| ≤ 2 · 10<sup>-4</sup>**

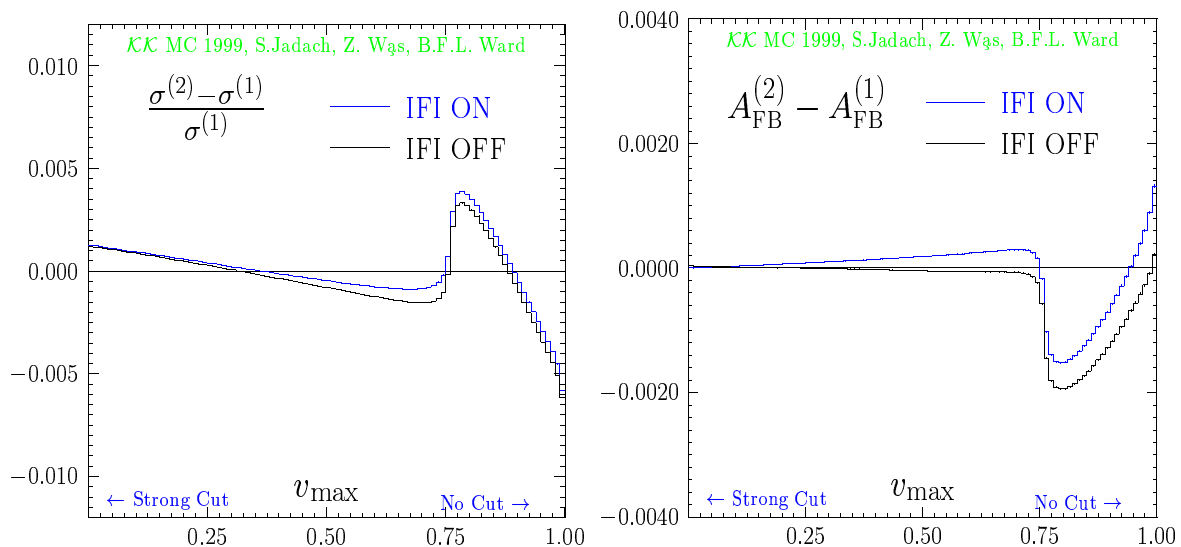
**PP = neglected higher orders and subleadings**

Our basic estimate:  $PP = \frac{1}{2} \left\{ \mathcal{O}(\alpha^2)_{\text{CEEX}} - \mathcal{O}(\alpha^1)_{\text{CEEX}} \right\}$

$\mathcal{K}\mathcal{K}$  MC results for  $e^+e^- \rightarrow \mu^+\mu^-$  at 189GeV

Photon Energy Cut:  $v = 1 - s'/s < v_{\text{max}}$ ;  $s' = m_{f\bar{f}}^2$

Angular Cut:  $|\cos \theta| < 1$



**The above result imply for Physical Precision:**

$$\frac{\delta\sigma}{\sigma} \leq 0.2\%, \quad \delta A_{\text{FB}} \leq 0.1\%,$$

**even including radiative-return on-shell Z.**

**This is confirmed by more tests, see later.**

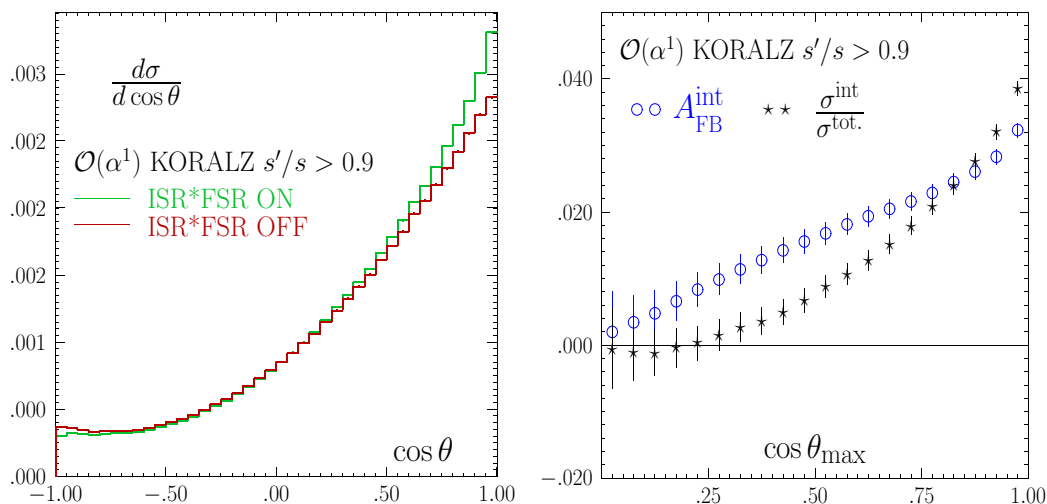


## $\mathcal{O}(\alpha^1)$ ISR ⊗ FSR from KORALZ

**KORALZ is the best reference and starting point for the ISR ⊗ FSR**

Angular distributions from KORALZ, pure  $\mathcal{O}(\alpha^1)$  (without exponentiation), were verified very precisely  $\sim 0.01\%$  using special analytical calculation,

**S.Jadach Z.Wąs, Phys. Rev. D41, 1425 (1990).**



Results above are for  $e^+e^- \rightarrow \mu^+\mu^-$  at  $\sqrt{s}=189\text{GeV}$ .

The energy cut is on  $s'/s$ , where  $s' = m_{f\bar{f}}^2$ .

The angular cut is  $|\cos\theta| < \cos\theta_{max}$ .

Scattering angle is  $\theta = \theta^\bullet$ ,

defined in Phys. Rev. D41, 1425 (1990)

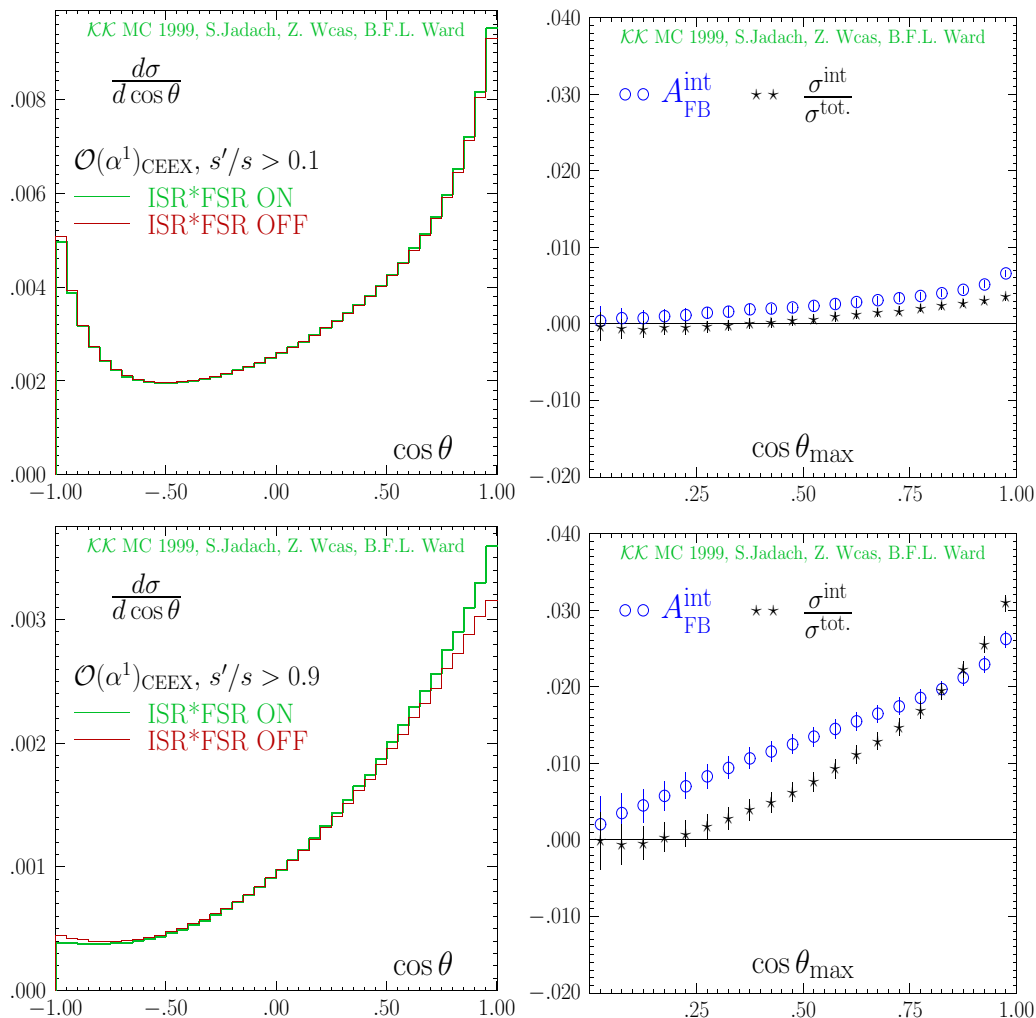
## Cut-off Dependence

$\mathcal{K}\mathcal{K}$  MC results for  $e^+e^- \rightarrow \mu^+\mu^-$  at 189GeV

Photon Energy Cut is on  $s' = m_{f\bar{f}}^2$

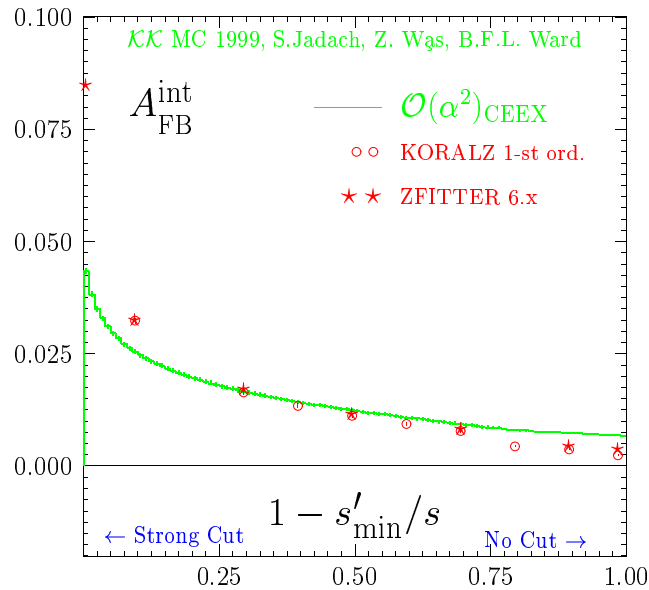
Angular Cut is  $|\cos \theta| < \cos \theta_{\max}$

Angle  $\theta = \theta^\bullet$  defined in Phys.Rev.D41,1425 (1990)

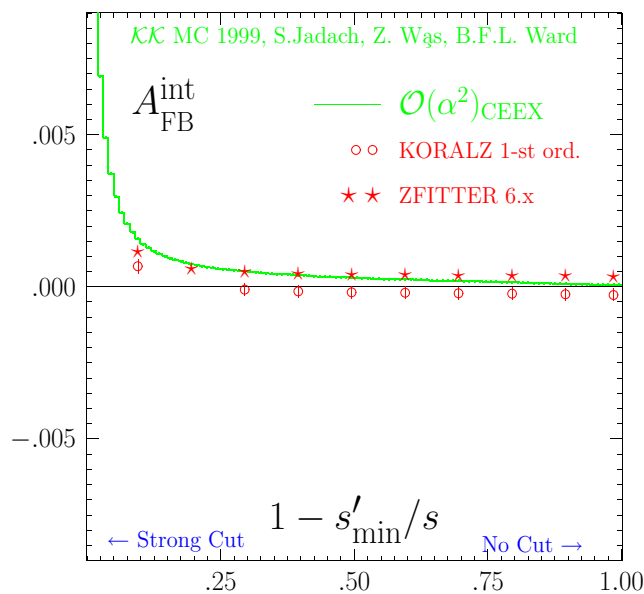


## $s'$ -cut dependence of $\delta A_{FB}$ . No $\theta$ -cut

At 189GeV the interference corr.  $\delta A_{FB}$  is 2%-5%:



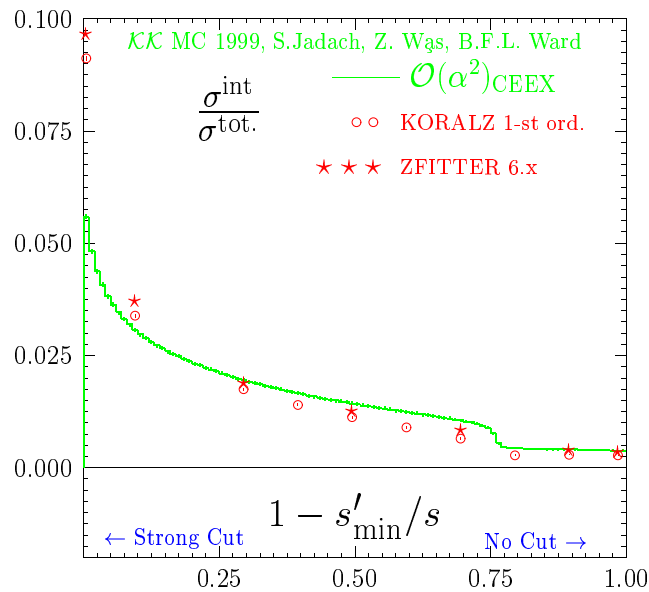
At  $\sqrt{s} = M_Z$  the effect is suppressed  $\delta A_{FB} < 0.1\%$ :



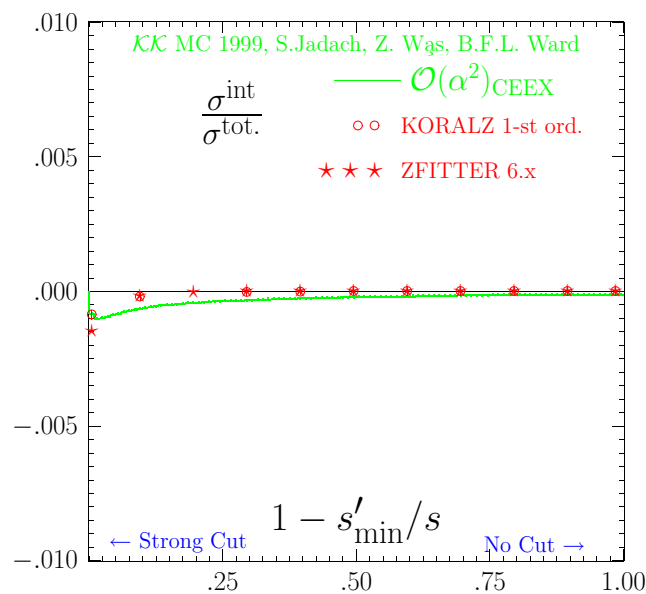
The effect increases strongly with the photon energy cut.

## $s'$ -cut dependence of $\delta\sigma$ , No $\theta$ -cut

At 189GeV interference eff.  $\delta\sigma$  is 1%-3%, no  $\cos\theta$ -cut:



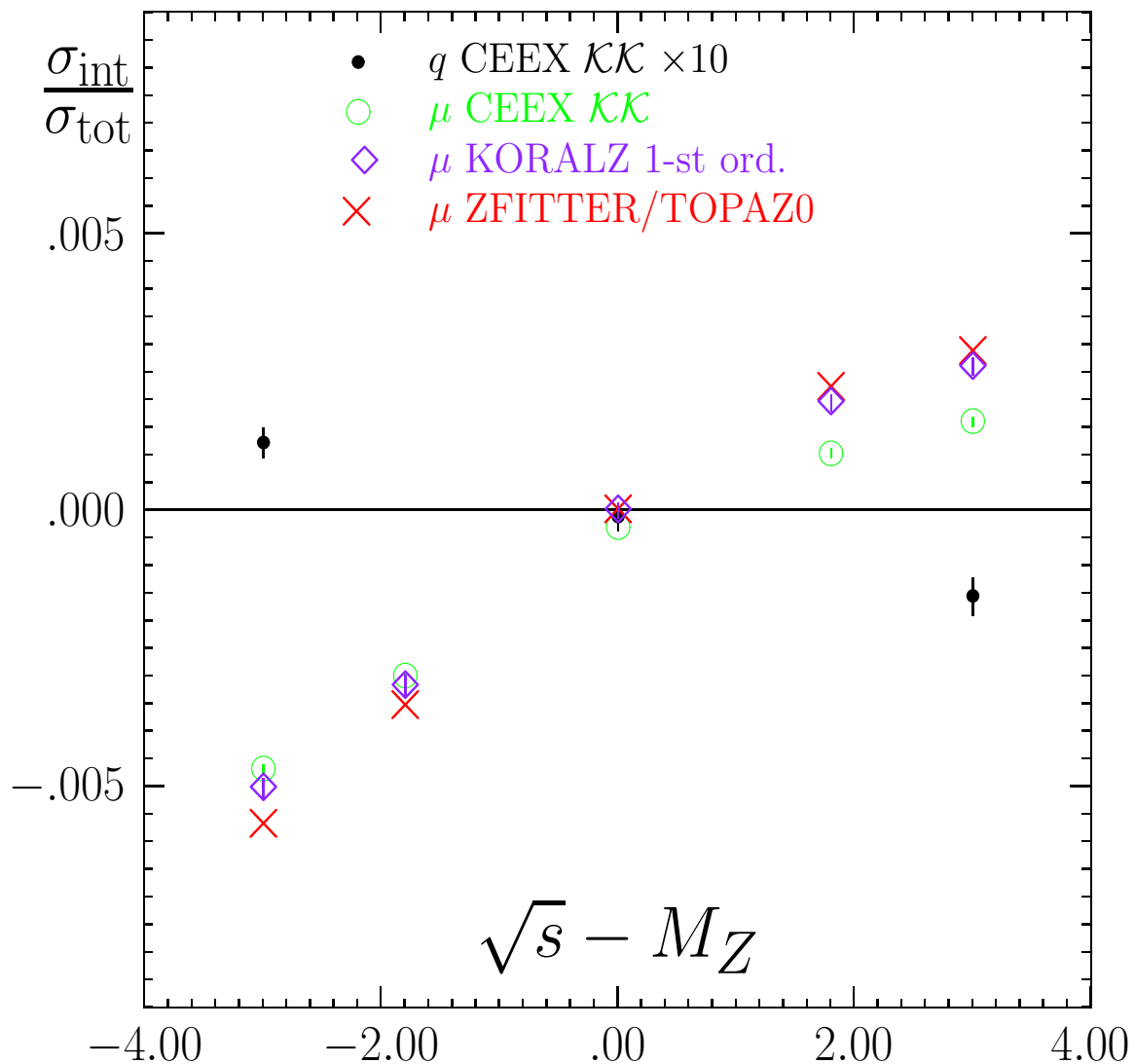
At  $\sqrt{s} = M_Z$  the effect is suppressed  $\delta\sigma < 0.02\%$ :



The effect increases strongly with the photon energy cut.

**Back on Z peak**

$\mathcal{K}\mathcal{K}$  MC agrees with KORALZ and semi-analytical programs and provides hint about higher orders.



## $\mathcal{K}\mathcal{K}$ Monte Carlo and KORALZ answers on ISR $\otimes$ FSR Interference

- For typical exp. energy cut 0.3 ISR  $\otimes$  FSR int. is about 1.5% in  $\sigma_{tot}$  and  $A_{FB}$ .
- For energy cut 0.1 it is twice bigger.
- The cut  $|\cos\theta| < 0.9$  makes it 25% smaller.
- The  $\mathcal{O}(\alpha^1)$  ISR  $\otimes$  FSR int. is under total control using KORALZ and  $\mathcal{K}\mathcal{K}$  Monte Carlo for arbitrary cuts.
- Effects beyond  $\mathcal{O}(\alpha^1)$  are negligible, ( $< 20\%$  of  $\mathcal{O}(\alpha^1)$ ), except when energy cut is stronger than 0.1.
- ISR  $\otimes$  FSR int. at Z radiative return is very small, as expected.
- Change from  $s'$  to  $Q^2$ -propagator in energy cut has no effect.

# Absolute prediction for $\sigma$ and $A_{FB}$

**Process:**  $e^-e^+ \rightarrow f\bar{f}$ ,  $f = \mu^-$ , at 189GeV.

**Energy cut:**  $v < v_{\max}$ , where  $v = 1 - s'/s$ ,  $s' = M_{f\bar{f}}^2$ .

**Scattering angle for  $A_{FB}$  is  $\theta = \theta^\bullet$ . No cut in  $\theta$ . E-W corr. in  $\mathcal{K}\mathcal{K}$  according to DIZET 6.x.  $\mathcal{O}(\alpha^3)_{LL}$  EEX3 matrix element in  $\mathcal{K}\mathcal{K}$  MC (No ISR $\otimes$ FSR interf.)**

**$\mathcal{K}\mathcal{K}$ sem is semianalytical part of  $\mathcal{K}\mathcal{K}$ .** (Angle  $\theta^\bullet$  is from Phys. Rev. D41, 1425 (1990).)

$v_{\max}$	$\mathcal{K}\mathcal{K}$ sem Refer.	$\mathcal{O}(\alpha^3)_{\text{EEX3}}$	$\mathcal{O}(\alpha^2)_{\text{CEEX intOFF}}$	$\mathcal{O}(\alpha^2)_{\text{CEEX}}$	KORALZ	KORALZ Interf.
$\sigma(v_{\max})$ [pb], $\mathcal{K}\mathcal{K}$ M.C. and KORALZ 1-st order						
0.01	$1.6712 \pm 0.0000$	$1.6687 \pm 0.0020$	$1.6669 \pm 0.0020$	$1.7657 \pm 0.0024$	$0.9639 \pm 0.0009$	$0.0912 \pm 0.0005$
0.10	$2.5198 \pm 0.0000$	$2.5164 \pm 0.0023$	$2.5147 \pm 0.0023$	$2.5943 \pm 0.0027$	$2.1919 \pm 0.0010$	$0.0339 \pm 0.0004$
0.30	$3.0616 \pm 0.0000$	$3.0565 \pm 0.0024$	$3.0578 \pm 0.0024$	$3.1183 \pm 0.0029$	$2.7690 \pm 0.0010$	$0.0175 \pm 0.0003$
0.50	$3.3747 \pm 0.0000$	$3.3682 \pm 0.0025$	$3.3740 \pm 0.0025$	$3.4219 \pm 0.0029$	$3.0565 \pm 0.0010$	$0.0113 \pm 0.0003$
0.70	$3.7225 \pm 0.0000$	$3.7131 \pm 0.0025$	$3.7257 \pm 0.0025$	$3.7629 \pm 0.0030$	$3.3649 \pm 0.0010$	$0.0065 \pm 0.0003$
0.90	$7.1434 \pm 0.0000$	$7.0904 \pm 0.0024$	$7.1538 \pm 0.0024$	$7.1795 \pm 0.0029$	$6.3558 \pm 0.0010$	$0.0029 \pm 0.0001$
0.99	$7.6145 \pm 0.0000$	$7.5511 \pm 0.0024$	$7.6642 \pm 0.0024$	$7.6888 \pm 0.0029$	$6.7004 \pm 0.0010$	$0.0028 \pm 0.0001$
$A_{FB}(v_{\max})$ , $\mathcal{K}\mathcal{K}$ M.C. and KORALZ 1-st order						
0.01	$0.5654 \pm 0.0000$	$0.5650 \pm 0.0014$	$0.5650 \pm 0.0014$	$0.6111 \pm 0.0016$	$0.5765 \pm 0.0013$	$0.1201 \pm 0.0013$
0.10	$0.5664 \pm 0.0000$	$0.5660 \pm 0.0011$	$0.5660 \pm 0.0011$	$0.5922 \pm 0.0012$	$0.5784 \pm 0.0006$	$0.0324 \pm 0.0006$
0.30	$0.5692 \pm 0.0000$	$0.5687 \pm 0.0009$	$0.5686 \pm 0.0009$	$0.5855 \pm 0.0011$	$0.5818 \pm 0.0005$	$0.0164 \pm 0.0005$
0.50	$0.5744 \pm 0.0000$	$0.5738 \pm 0.0009$	$0.5737 \pm 0.0009$	$0.5862 \pm 0.0010$	$0.5868 \pm 0.0005$	$0.0112 \pm 0.0005$
0.70	$0.5864 \pm 0.0000$	$0.5852 \pm 0.0008$	$0.5853 \pm 0.0008$	$0.5944 \pm 0.0009$	$0.5972 \pm 0.0004$	$0.0078 \pm 0.0004$
0.90	$0.3105 \pm 0.0000$	$0.3115 \pm 0.0004$	$0.3108 \pm 0.0004$	$0.3177 \pm 0.0005$	$0.3260 \pm 0.0002$	$0.0037 \pm 0.0002$
0.99	$0.2851 \pm 0.0000$	$0.2867 \pm 0.0004$	$0.2844 \pm 0.0004$	$0.2903 \pm 0.0004$	$0.3039 \pm 0.0002$	$0.0024 \pm 0.0002$

Process:  $e^-e^+ \rightarrow \mu^-\bar{\mu}^-$ , at 189GeV.

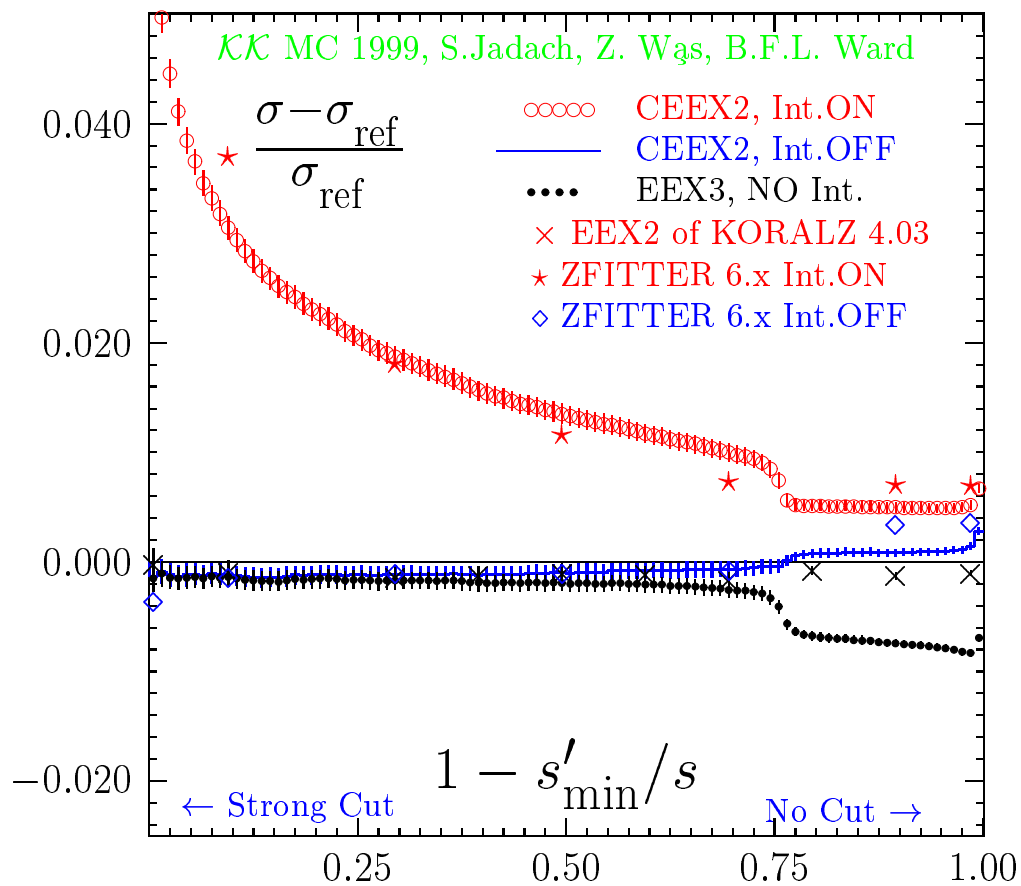
Energy cut:  $s' > s'_{\min}$ , where  $s' = M_{f\bar{f}}^2$ . No cut in  $\theta$ .

KORALZ/YFS3 version 4.03,  $\mathcal{O}(\alpha^2)_{\text{LL}}$ ,  $\text{ISR} \otimes \text{FSR}$  off.

$\text{ISR} \otimes \text{FSR}$  interf. on and off, wherever possible.

Reference  $\sigma_{\text{ref}}$  is from semi-analytical  $\mathcal{K}\mathcal{K}\text{SEM}$ ,

$\text{ISR} \otimes \text{FSR}$  off, up to  $\mathcal{O}(\alpha^3)_{\text{LL}}$ , JSW exponent.





Process:  $e^-e^+ \rightarrow \mu^-\bar{\mu}^-$ , at 189GeV.

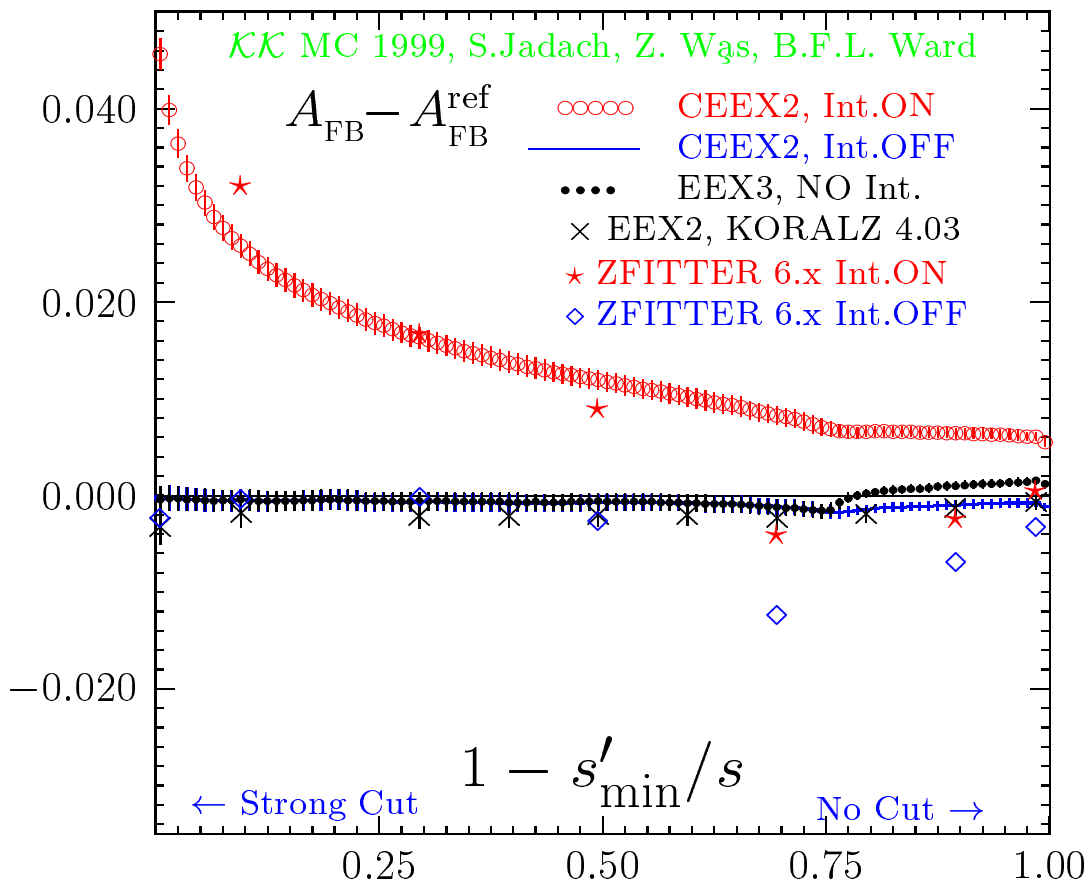
Energy cut:  $s' > s'_{\min}$ , where  $s' = M_{f\bar{f}}^2$ . No cut in  $\theta$ .

KORALZ/YFS3 version 4.03,  $\mathcal{O}(\alpha^2)_{LL}$ ,  $\text{ISR} \otimes \text{FSR}$  off.

$\text{ISR} \otimes \text{FSR}$  interf. on and off, wherever possible.

Reference  $A_{FB}^{\text{ref}}$  is from semi-analytical  $\mathcal{K}\mathcal{K}\text{SEM}$ ,

$\text{ISR} \otimes \text{FSR}$  off, up to  $\mathcal{O}(\alpha^3)_{LL}$ , JSW exponent.



## Summary of Part IV

- **Technical Precision**  $\frac{\delta\sigma}{\sigma} \leq 2 \cdot 10^{-4}$
- **Physical Precision:**  
 $\frac{\delta\sigma}{\sigma} \leq 0.2\%$  and  $\delta A_{FB} \leq 0.1\%$
- **ISR $\otimes$ FSR interf. under control at LEP2 energies**

**Comparison with our older MC's**

Feature	KORALB	KORALZ	$\mathcal{K}\mathcal{K}$ 4.13	$\mathcal{K}\mathcal{K}$ 2000+?
QED type	$\mathcal{O}(\alpha)$	EEX	CEEX, EEX	CEEX, EEX
CEEX(ISR+FSR)	none	none	$\{\alpha, \alpha L; \alpha^2 L^2, \alpha^2 L^1\}$	$\{\dots \alpha^2 L^1; \alpha^3 L^3\}$
EEX(ISR*FSR)	none	$\{\alpha, \alpha L, \alpha^2 L^2\}$	$\{\alpha, \alpha L, \alpha^2 L^2, \alpha^3 L^3\}$	$\{\dots \alpha^2 L^2, \alpha^3 L^3\}$
ISR-FSR int.	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha)$	$\{\alpha, \alpha L\}_{\text{CEEX}}$	$\{\alpha, \alpha L\}_{\text{CEEX}}$
Exact brems.	1 $\gamma$	1, 2coll. $\gamma$	1, 2, 3coll. $\gamma$	up to 3 $\gamma$
Electroweak	No Z-res.	DIZET 6.x	DIZET 6.x	New version?
Beam polar.	long+trans.	longit.	long+trans.	long+trans.
$\mathcal{T}$ polar.	long+trans.	longit.	long+trans.	long+trans.
Hadronization	—	JETSET	JETSET	PYTHIA
$\mathcal{T}$ decay	TAUOLA	TAUOLA	TAUOLA	TAUOLA
Inclusive mode	—	No	Yes	Yes
Beamstrahlung	—	No	Yes	Yes
Beam spread	—	No	Yes	Yes
$\nu\nu$ channel	—	Yes	No	Yes
$ee$ channel	—	No	No	Yes
$tt$ channel	—	No	No	yes?
$WW$ channel	—	No	No	yes?

## Conclusions

$\mathcal{K}\mathcal{K}$  MC based on CEEEX offers clear upgrade path for the MC event generators to Linear Collider era. CEEEX is firmly based on spin amplitudes.

The main profits are:

- Interferences  $\text{ISR} \otimes \text{FSR}$  included and under firm control
- All kind of coherence effects, including narrow resonances.
- Complete treatment of spin (also transverse) for beams and final (unstable) fermions.
- Exact M.E. for 2 high hard photons (3 photon pending).
- Future extension to all-angle second order Bhabha is possible.

**First official version 4.13 of  $\mathcal{K}\mathcal{K}$  MC available. It works for fermion pair production at LEP, LC's,  $\tau$  and  $b$  factories,  $\mu$ -colliders.**

Check <http://home.cern.ch/jadach> for slides, programs.