Coherent Exclusive EXponentiation For e^+e^- Collision at LEP and LCs

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Outline:

- **Introduction**
- **Kleiss-Stirling+GPS spinor technique**
- **CEEX Amplitudes and differ. cross sections**
- **Example numerical results**
- **Conclusions**

Papers by S. Jadach, B.F.L. Ward and Z. Wąs:

CERN-TH/98-253, Phys. Lett. B449 (1999) 97,

CERN-TH/98-235, subm. to EPJ,

DESY-99-106 (CERN-TH/99-235), subm. to Comput. Phys. Commun.

The problem

Deficiencies of YFS **Exclusive Exponentiation** (EEX) technique, as implemented in KORALZ/YFS3, BHLUMI, BHWIDE, KORALW:

- Lack of the ISR/FSR or Up/Down interferences
  (except BHWIDE, but difficult to upgrade...)
- Simplified treatment of spin, neglected transverse polarization
  (except KORALB, but no exponentiation...)
- Approximate matrix element for 2 and 3 hard large p_T photons.
  (Important for searches.)

The CEEX solution

- ISR/FSR interferences are included in a natural way.
  Spin amplitudes for ISR and FSR summed squared numerically.
  (The BHWIDE approach gets cumbersome beyond first order)
- Exact treatment of fermion spin polarizations (transv.&longitudinal).
  Numerical Wigner rotations of spin density matrices available.
  (Necessary for interfacing with decay M.C. simulating fermion decays).
- Exact matrix element for 2 and 3 and more photons using
  Kleiss-Stirling (KS) spinor technique.

Comparison with our older MC's

Feature	KORALB	KORALZ	$\mathcal{KK}$ now	$\mathcal{KK}$ 2000
QED type	$\mathcal{O}(\alpha)$	EEX	CEEX, EEX	CEEX, EEX
CEEX(ISR+FSR)	none	none	$\{\alpha, \alpha L; \alpha^2 L^2, \alpha^2 L^1\}$	$\{\ldots\alpha^2 L^1; \alpha^3 L^3\}$
EEX(ISR*FSR)	none	$\{\alpha, \alpha L, \alpha^2 L^2\}$	$\{\alpha, \alpha L, \alpha^2 L^2, \alpha^3 L^3\}$	$\{\ldots\alpha^2 L^2, \alpha^3 L^3\}$
ISR-FSR int.	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha)$	$\{\alpha, \alpha L\}_{\rm CEEX}$	$\{\alpha, \alpha L\}_{\rm CEEX}$
Exact bremss.	1 γ	1, 2coll. γ	1, 2, 3coll. γ	up to 3 γ
El-Weak	No Z-res.	DIZET 6.x	DIZET 6.x	YES
Beam polar.	long+trans.	longit.	long+trans.	long+trans.
τ polar.	long+trans.	longit.	long+trans.	long+trans.
Hadronization	—	JETSET	JETSET	PYTHIA
τ decay	TAUOLA	TAUOLA	TAUOLA	TAUOLA
Inclusive mode	—	No	Yes	Yes
Beamstrahlung	—	No	Yes	Yes
beam spread	—	No	Yes	Yes
$\nu\nu$ channel	—	Yes	No	Yes
ee channel	—	No	No	Yes
tt channel	—	No	No	yes?
WW channel	—	No	No	yes?

KS-GPS Technique

Kleiss-Stirling spinors

Any MASSLESS spinors is transformed out of two

basic constant **spinors** $\mathfrak{u}_\pm(\zeta)$, $\zeta^2 = 0$: $u_\lambda(p) = \frac{1}{\sqrt{2p\cdot\zeta}}\ \not{p}\mathfrak{u}_{-\lambda}(\zeta)$,

where $\mathfrak{u}_+(\zeta) = \not{\eta}\mathfrak{u}_-(\zeta),\ \eta^2 = -1,\ (\eta\zeta) = 0.$

$\not{\zeta}\mathfrak{u}_\lambda(\zeta) = 0, \omega_\lambda \mathfrak{u}_\lambda(\zeta) = \mathfrak{u}_\lambda(\zeta), \mathfrak{u}_\lambda(\zeta)\bar{\mathfrak{u}}_\lambda(\zeta) = \not{\zeta}\omega_\lambda,$

$\not{p}u_\lambda(p) = 0, \omega_\lambda u_\lambda(p) = u_\lambda(p), u_\lambda(p)\bar{u}_\lambda(p) = \not{p}\omega_\lambda$, **and** $\omega_\lambda = \frac{1}{2}(1+\lambda\gamma_5)$.

Spinors for the MASSIVE hal spin fermion with 4-momentum p **(**$p^2 = m^2$**) and spin projection** $\lambda/2$ **are defined in terms of massless spinors:**

$u(p,\lambda) = u_\lambda(p_\zeta) + \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_{-\lambda}(\zeta)$ **and** $v(p,\lambda) = u_{-\lambda}(p_\zeta) - \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_\lambda(\zeta)$,

where $p_{i\zeta} \equiv \hat{p}_i \equiv p_i - \zeta\ m_i^2/(2\zeta p_i)$.

We exploit very often the standard completeness relations:

$$\not{p} + m = \sum_\lambda u(p,\lambda)\bar{u}(p,\lambda), \quad \not{p} - m = \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda)$$

KS-GPS Technique

Toolbox for Kleiss-Stirling spinors

The *inner product* of the two *massless* spinors is defined as follows:

$$s_+(p_1,p_2) \equiv \bar{u}_+(p_1)u_-(p_2), \quad s_-(p_1,p_2) \equiv \bar{u}_-(p_1)u_+(p_2) = -(s_+(p_1,p_2))^*.$$

In any reference frame it can be evaluated using the Kleiss-Stirling expression:

$$s_+(p,q) = 2\ (2p\zeta)^{-1/2}\ (2q\zeta)^{-1/2}\ \left[(p\zeta)(q\eta) - (p\eta)(q\zeta) - i\epsilon_{\mu\nu\rho\sigma}\zeta^\mu\eta^\nu p^\rho q^\sigma\right]$$

For example, if in LAB frame $\zeta = (1,1,0,0)$ and $\eta = (0,0,1,0)$, then in this frame:

$$s_+(p,q) = -(q^2+iq^3)\sqrt{(p^0-p^1)/(q^0-q^1)} + (p^2+ip^3)\sqrt{(q^0-q^1)/(p^0-p^1)}.$$

The *inner product* of the two *massive* spinors is:

$$\bar{u}(p_1,\lambda_1)u(p_2,\lambda_2) = S(p_1,m_1,\lambda_1,p_2,m_2,\lambda_2),$$

$$\bar{u}(p_1,\lambda_1)v(p_2,\lambda_2) = S(p_1,m_1,\lambda_1,p_2,-m_2,-\lambda_2),$$

$$\bar{v}(p_1,\lambda_1)u(p_2,\lambda_2) = S(p_1,-m_1,-\lambda_1,p_2,m_2,\lambda_2),$$

$$\bar{v}(p_1,\lambda_1)v(p_2,\lambda_2) = S(p_1,-m_1,-\lambda_1,p_2,-m_2,-\lambda_2),$$

where

$$S(p_1,m_1,\lambda_1,p_2,m_2,\lambda_2) = \delta_{\lambda_1,-\lambda_2}\, s_{\lambda_1}(p_{1\zeta},p_{2\zeta}) + \delta_{\lambda_1,\lambda_2}\left(m_1\sqrt{\frac{2\zeta p_2}{2\zeta p_1}} + m_2\sqrt{\frac{2\zeta p_1}{2\zeta p_2}}\right)$$

and $p_{i\zeta} \equiv \hat{p_i} \equiv p_i - \zeta\, m_i^2/(2\zeta p_i)$

KS-GPS Technique

Why do we need GPS?

GOODIES of the Wigner-Jacob-Wick spinology we know and love:

Elegant definition of spin state and scattering matrix element

$$|p,n>=T(L(p))|\overset{\circ}{p},n>,\quad \mathcal{M}_{nm}=<f|S|i>=<q,m|S|p,n>$$

Simple transformation under rotation using Wigner $\mathcal{D}^s$-matrices

$$T(R)\ |\overset{\circ}{p},n>=\sum_{n'}|\overset{\circ}{p},n'>\mathcal{D}^s_{n'n}(R)$$

Well defined (so called) Wigner-Wick rotation

$$\mathcal{M}_{nm}=\sum_{m',n'}\mathcal{M}_{n'm'}\mathcal{D}^s_{n'n}(R^W_i)\mathcal{D}^{s+}_{m'm}(R^W_f)$$

where R^{W} is rotation reflecting directly the change of the quantization axes (in rest frame of the massive particle) or is "generated" by the general Lorentz transformation, as in the paper of Wick (Ann. Phys. 18 (1962) 65) for the Jacob-Wick helicity states.

How to combine the above with Weyl spinor techniques? $\longrightarrow$ GPS rules.

WHY BOTHER? For unstable fermions like τ production and decay spin amplitudes are calculated using completely different methods and with different quantization axes. We need to know very precisely the spin quantization axes for KS spinors in the M.C. event generation, especially for unstable fermions.

KS-GPS Technique

GPS – preliminaria

The necessary condition for validity of the usual Wigner-Jacob-Wick technology (including standard Clebsch-Gordan coefficients etc.) is the textbook phase relation:

$$(J_x \pm i J_y)|m>= [(s \mp m)(s \pm m + 1)]^{1/2}| \pm m>$$

(National Bureau of Standards 1951)

equivalent to a condition with the prosaic rotation by $+\pi$ around y-axis

$$\exp(-\tfrac{1}{2}\, i\pi\; J_y)|+>=|->,$$ **in our $s = 1/2$ case.**

Take Weyl representation. Let us define Primary Reference Frame PRF where $\zeta = (1,0,0,-1)$ and $\eta = (0,1,0,0)$, (in general PRF$\neq$LAB). If the massive fermion is at rest in the PRF then for KS spinors $u(p,\pm)$ and $v(p,\pm)$ the above phase relation is fulfilled, i.e., spin is quantized using z-axis; the x and y axes are correctly positioned!

In the general case of the fermion in flight the GPS rule will tell us where the quantization z-axis is, and also the x and y axes relevant for the relative phases of $|\pm>$. See next slide.

KS-GPS Technique

GPS stands for Global Positioning of Spin

The rules for determining all three spin quantization axes for $u(p,\pm)$ and $v(p,\pm)$ defi ned with KS method for Weyl representation are the following:

- **In the rest frame of the fermion, take the z-axis along $-\vec{\zeta}$.**
- **Place the x-axis in the plane defined by the z-axis from the previous point and the vector $\vec{\eta}$, in the same half-plane as $\vec{\eta}$.**
- **With the y-axis, complete the right-handed system of coordinates.**

We call the above rules the GPS rules, and we shall call the Spin Quantization Reference Frame (SQRF) determined by the above GPS rules the GPS frame of the fermion.

The formal proofs of the above rules is in CERN-TH-98-235. It amounts to showing that in GPS fermion rest frame the two *constant basic* spinors $\mathfrak{u}_{\pm}(\zeta)$, up to a real constant, are the same (have the same components) as in PRF.

KS-GPS Technique

GPS at work: polarized τ decay

Suppose, that for $\tau(p) \to X$ decay, using Feynman rules and classic "Dirac alchemy", including $u(p,s)\,\bar{u}(p,s) = \frac{1}{2}(1+\gamma_5 \not{s})\,(\not{p}+m)$, we obtain (def. polarimeter vector h):

$$d\Gamma^{pol.}_{class.}(s) = d\Gamma^{unpol.}(q_1 ... q_n)\,\big(1 + s \cdot h(q_1 ... q_n)\big),$$ where $h \cdot p = 0$.

On the other hand, using KS spinors, we calculate spin amplitudes $\mathcal{N}_\mu$ for decay process.

Remembering the textbook definition of spin density matrix we can write:

$$d\Gamma^{pol.}_{KS}(s) = \sum_{\mu,\bar{\mu}=\pm 1/2} \rho_{\mu\bar{\mu}} \mathcal{N}_\mu \mathcal{N}^*_{\bar{\mu}} d\Phi_{dec.}$$ where ρ is the spin density matrix.

HOW TO RELATE $\mathcal{N}_\mu$ AND h_μ IN THE TWO CALCULATION METHODS?

Seems easy: from textbook relation $\rho = \sum_{k=0}^{3} \sigma^k \hat{s}^k$ where $\hat{s} = (1, \vec{s})$ we get:

$$d\Gamma^{pol.}(s) = \hat{s}^a \left[\sigma^a_{\mu\bar{\mu}} \mathcal{N}_\mu \mathcal{N}^*_{\bar{\mu}}\right] d\Phi_{dec.} = \hat{s}^a \hat{h}_a d\Gamma^{unpol.}$$ and therefore we IDENTIFY:

$$\hat{h}_a = (1, \vec{h}(q_1 ... q_n)) = \sum_{\rho\bar{\rho}} \sigma^a_{\rho\bar{\rho}} \mathcal{N}_\rho \mathcal{N}^*_{\bar{\rho}} \Big/ \Big(\sum_\rho \mathcal{N}_\rho \mathcal{N}^*_\rho\Big)$$ where σ^a are Pauli matrices. All the above was in the τ rest frame. **WHICH FRAME**?

ONLY in the GPS rest frame! Why? In other frame we would be forced to replace Pauli matrices with something else; **i.e. with non-standard analog of Clebsch-Gordan coefficients, because the phase relation $(J_x + iJ_y)|-> = |+>$ holds for our spin states only in the GPS frame (note that Pauli σ's in $\rho = \sum_{k=0}^{3} \sigma^k \hat{s}^k$ are here just Clebsch-Gordan coefficients for $D^{1/2} \otimes D^{1/2}$).**

KS-GPS Technique

GPS at work: Wigner/Wick rotation

For the τ production and decay process

$$e^-(p_1,\lambda_1)+e^+(p_2,\lambda_2)\to\tau^-(q_1,\mu_1)+\tau^+(q_2,\mu_2),\ \tau^\pm\to X^\pm,$$

following the same lines, we may write the spin amplitudes:

$$\mathcal{M}_{\lambda_1\lambda_2}=\sum_{\mu_1\mu_2}\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}\mathcal{N}_{\mu_1}\mathcal{N}'_{\mu_2}$$

and the polarized differential cross-section:

$$d\sigma=\sum_{\mu_i\bar\mu_i\lambda_i\bar\lambda_i}\rho_{\lambda_1\bar\lambda_1}\rho'_{\lambda_2\bar\lambda_2}\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}\mathcal{M}^*_{\bar\lambda_1\bar\lambda_2\bar\mu_1\bar\mu_2}d\Phi_{\rm prod.}\ \mathcal{N}_{\mu_1}\mathcal{N}^*_{\bar\mu_1}d\Phi_\tau\mathcal{N}'_{\mu_2}\mathcal{N}'^*_{\bar\mu_2}d\Phi'_\tau$$

In terms of beam polarizations $\hat s$ and decay polarimeter vectors $\hat h$ we have:

$$d\sigma=\sum_{abcd}\hat s^a\hat s'^b\ R^{cd}_{ab}\ d\sigma^{\rm prod.}_{\rm unpol.}\ \hat h_c\ d\Gamma_{\rm unpol.}\ \hat h'_d\ d\Gamma'_{\rm unpol.}$$ where

$$R^{cd}_{ab}=\frac{\sum_{\mu_i\bar\mu_i\lambda_i\bar\lambda_i}\sigma^a_{\lambda_1\bar\lambda_1}\sigma^b_{\lambda_2\bar\lambda_2}\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}\mathcal{M}^*_{\bar\lambda_1\bar\lambda_2\bar\mu_1\bar\mu_2}\sigma^c_{\bar\mu_1\mu_1}\sigma^d_{\bar\mu_2\mu_2}}{\sum_{\mu_i\bar\mu_i\lambda_i\bar\lambda_i}|\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}|^2}$$

The above Wigner-Jacob-Wick spin technology is used in KORALB MC.

If the production amplitudes $\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}$ are calculated using KS spinors then we may continue to use the above "spinology" **only if we apply it in the GPS** rest frames of all four fermions! Consequently, we need Wigner/Wick rotations. Why? See next slide.

KS-GPS Technique

GPS at work: Wigner/Wick rotation

For $e^{\pm}$ beams we know polarization vectors $\vec{s}_{\pm}$ in certain well defi ned rest frames BRF$_{\pm}$ of the $e^{\pm}$ (from the machine people). In other words, we know precisely the Lorentz transformations $L_{\pm}^{\rm BRF}$: BRF$_{\pm} \longrightarrow$ LAB. On the other hand, from GPS rules, we know transformations $L_{\pm}^{\rm GPS}$: GPS($e^{\pm}$) $\longrightarrow$ LAB. If we use KS spinors for the scattering spin amplitudes, then we have to apply the Wigner/Wick rotation $R_{\pm}^{W} = L_{\pm}^{\rm GPS}(L_{\pm}^{\rm BRF})^{-1}$, i.e. rotate $\vec{s}_{\pm}$ from BRF$_{\pm}$ to GPS($e^{\pm}$). (In practice this is a rotation around the beam axis).

For $\tau^{\pm}$ we avoid explicit Wigner/Wick rotation, if the decays of polarized τ's are done (simulated in MC) precisely in GPS($\tau^{\pm}$) frames.

We may need Wigner/Wick rotation if we want to compare spin amplitudes from KS method and some other method. Next slide shows such a comparison for the spin correlation tensor R_{00}^{cd} from two methods: (1) KS spinors, (2) with Jacob-Wick (JW) helicity states (Jadach Wąs 1984, Tsai 1971); in fact the variant of JW called JW2.

Numerical test of GPS

```
 ++++++++++++++++++++++++++++++++++++++++ Born spin amplitudes ++++++++++++++++++++++++++++++++
     { 1 1 1 1}  { 1 1-1 1}  { 1 1 1-1}  { 1 1-1-1}
     { 1-1 1 1}  { 1-1-1 1}  { 1-1 1-1}  { 1-1-1-1}
     {-1 1 1 1}  {-1 1-1 1}  {-1 1 1-1}  {-1 1-1-1}
     {-1-1 1 1}  {-1-1-1 1}  {-1-1 1-1}  {-1-1-1-1}
[    .000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
[    .000000   -.454758] [   -.696227   -.001005] [   -.280666   -.001005] [    .000000    .454758]
[    .000000    .454758] [   -.280666    .001005] [   -.696227    .001005] [    .000000   -.454758]
[    .000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000

###################################### R-matrix GPS fram ######################################
[   1.000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
[    .000000    .000000] [   -.023328    .000000] [    .000855    .000000] [    .001871    .000000]
[    .000000    .000000] [   -.000855    .000000] [    .823264    .000000] [   -.386811    .000000]
[    .000000    .000000] [    .001871    .000000] [    .386811    .000000] [   -.153408    .000000]

###################################### R-matrix JacWick  ######################################
[   1.000000    .000000] [    .000000    .000000] [    .000000    .000000] [    .000000    .000000]
[    .000000    .000000] [    .199012    .000000] [    .000000    .000000] [   -.375031    .000000]
[    .000000    .000000] [    .000000    .000000] [    .023418    .000000] [    .000000    .000000]
[    .000000    .000000] [    .375031    .000000] [    .000000    .000000] [   -.824406    .000000]

********************    R analytical     ********************
[     1.000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[      .000000000000] [      .199012228374] [      .000000000000] [     -.375031393988]
[      .000000000000] [      .000000000000] [      .023418353587] [      .000000000000]
[      .000000000000] [      .375031393988] [      .000000000000] [     -.824406125213]

********************    R Difference     ********************
[      .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[      .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[      .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
[      .000000000000] [      .000000000000] [      .000000000000] [      .000000000000]
cos(theta),sin(theta)=      -.884680271149      .466198260229
amfin,sqrt(s)=      1.777000000000     4.000000000000
```

Numerical illustration of Wigner-Wick rotation for one τ-pair production event at $\sqrt{s}=$4 GeV, $m_\tau = 1.777$ GeV, $\cos\theta = -0.884680271149$. In the upper part we print the complex spin amplitudes $\mathcal{M}^{\rm prod}_{\lambda_1\lambda_2\mu_1\mu_2}$, with the numbering style $\{\lambda_1\lambda_2\mu_1\mu_2\}$ from KS spinors. Then, we show the $R^{cd}_{00},\ c,d=0\dots3$, correlation matrix as calculated in the GPS frame, and after a (double) Wigner-Wick rotation to JW2 τ rest frames. The result marked “analytical” is obtained analytically (Jadach Wąs 1984) also in the JW2 frames. The difference of the two, in the JW2 frame, is zero.

Appendix A: Wigner-Wick rotations, collection of formulae

STATES in Hilbert space transform under rotation as follows:

$$|\overset{\circ}{p},\mu>_R=\sum_{\mu'}|\overset{\circ}{p},\mu'>\mathcal{D}^{1/2}_{\mu'\mu}(R),\quad <\overset{\circ}{p},\mu|_R=\sum_{\mu'}\mathcal{D}^{1/2\,\dagger}_{\mu\mu'}(R)<\overset{\circ}{p},\mu'|.$$

Consequently, **SPIN AMPLITUDES** transform under four different Wigner rotations:

$$\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}=\sum_{\lambda_i\mu_i}\mathcal{M}_{\lambda'_1\lambda'_2\mu'_1\mu'_2}\mathcal{D}^{1/2}_{\lambda'_1\lambda_1}(R^i_1)\mathcal{D}^{1/2}_{\lambda'_2\lambda_2}(R^i_2)\mathcal{D}^{1/2\,\dagger}_{\mu_1\mu'_1}(R^f_1)\mathcal{D}^{1/2\,\dagger}_{\mu_2\mu'_2}(R^f_2),$$

Final an initial state spin **DENSITY MATRICES** transform by Wigner rotation in the same way:

$$(\rho_{\mu\bar{\mu}})_R=\sum_{\mu'\bar{\mu}'}\mathcal{D}^{1/2\,\dagger}_{\mu\mu'}(R)\;\rho_{\mu'\bar{\mu}'}\;\mathcal{D}^{1/2}_{\bar{\mu}'\bar{\mu}}(R)$$

The above transformation induces through $s^k=\mathrm{Tr}(\rho\sigma^k)$ the ordinary rotation transformation of the spin **POLARIZATION VECTOR** (as it should!) e.g. $\vec{s}$ transforms as a contravariant vector:

$$(s^k)_R=\sum_{k'=1}^{3}R^k{}_{k'}\,s^{k'},$$

On the other hand the h_a POLARIMETER VECTOR (also related to spin amplitudes with Pauli matrices) transforms with transposed rotation matrix, e.g. as a covariant vector:

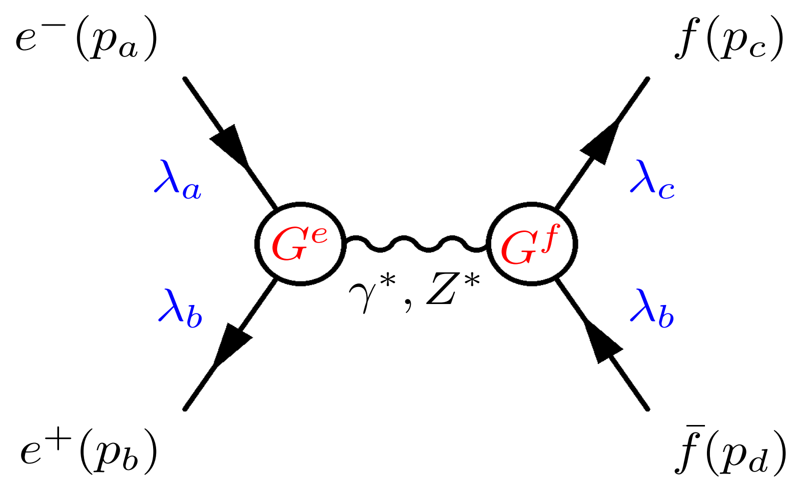
$$(h_a)_R=\sum_{a'=1}^{3}h_{a'}R^{a'}{}_a$$

Finally the full spin **CORRELATION TENSOR** transforms as follows:

$$(R^{cd}_{ab})_R=\sum_{a'b'c'd'=1}^{3}R^{c'd'}_{a'b'}\;(R^i_1)^{a'}{}_a\;(R^i_2)^{b'}{}_b\;(R^f_1)^c{}_{c'}\;(R^f_2)^d{}_{d'}.$$

Easy?!

GPS Appendix B: Born Spin Amplitudes, definition



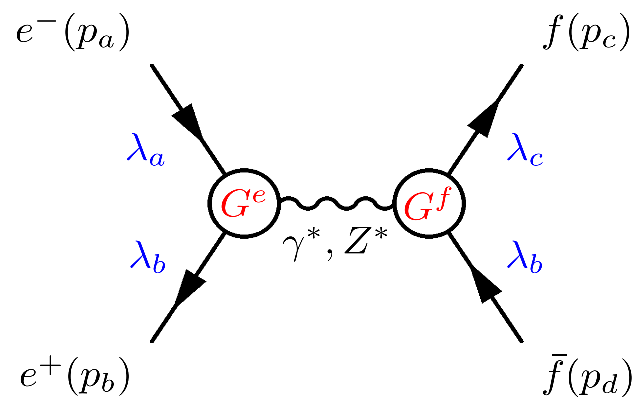
$$\mathfrak{B}\left(\begin{smallmatrix} p \\ \lambda \end{smallmatrix}; X\right) = \mathfrak{B}\left(\begin{smallmatrix} p_a & p_b & p_c & p_d \\ \lambda_a & \lambda_b & \lambda_c & \lambda_d \end{smallmatrix}; X\right) =$$

$$= 2ie^2 \sum_{B=\gamma,Z} \frac{\bar{v}(p_b,\lambda_b)\gamma^\mu G^{e,B} u(p_a,\lambda_a)\;\; \bar{u}(p_c,\lambda_c)\gamma_\mu G^{f,B} v(p_d,\lambda_d)}{X^2 - {M_B}^2 + i\Gamma_B X^2/M_B},$$

$$G^{e,B} = \sum_{\lambda=\pm} \frac{1}{2}(1+\lambda\gamma_5) g_\lambda^{e,B}, \quad G^{f,B} = \sum_{\lambda=\pm} \frac{1}{2}(1+\lambda\gamma_5) g_\lambda^{f,B},$$

KS-GPS Technique

GPS Appendix C: Born Spin Amplitudes in terms of KS spinors



$$\mathfrak{B}\begin{pmatrix} p_a & p_b & p_c & p_d \\ \lambda_a & \lambda_b & \lambda_c & \lambda_d \end{pmatrix}; X \Big) = ie^2 \sum_{B=\gamma,Z} \frac{\bar{v}(p_b,\lambda_b)\gamma^\mu G^{e,B} u(p_a,\lambda_a)\ \ \bar{u}(p_c,\lambda_c)\gamma_\mu G^{f,B} v(p_d,\lambda_d)}{X^2 - M_B{}^2 + i\Gamma_B X^2/M_B}$$

$$= 2ie^2 \sum_{B=\gamma,Z} \frac{\delta_{\lambda_a,-\lambda_b}\left[\, g^{e,B}_{\lambda_a} g^{f,B}_{-\lambda_a}\ T_{\lambda_c\lambda_a}\ T'_{\lambda_b\lambda_d} + g^{e,B}_{\lambda_a} g^{f,B}_{\lambda_a}\ U'_{\lambda_c\lambda_b}\ U_{\lambda_a\lambda_d}\right]}{X^2 - M_B{}^2 + i\Gamma_B X^2/M_B},$$

$$T_{\lambda_c\lambda_a} = \bar{u}(p_c,\lambda_c)u(p_a,\lambda_a) = S(p_c,m_c,\lambda_c,p_a,0,\lambda_a),$$

$$T'_{\lambda_b\lambda_d} = \bar{v}(p_b,\lambda_b)v(p_d,\lambda_d) = S(p_b,0,-\lambda_b,p_d,-m_d,-\lambda_d),$$

$$U'_{\lambda_c\lambda_b} = \bar{u}(p_c,\lambda_c)v(p_b,-\lambda_b) = S(p_c,m_c,\lambda_c,p_b,0,\lambda_b),$$

$$U_{\lambda_a\lambda_d} = \bar{u}(p_a,-\lambda_a)v(p_d,\lambda_d) = S(p_a,0,-\lambda_a,p_d,-m_d,-\lambda_d).$$

KS-GPS Technique

Conclusions for Part II

- Our primary aim is to use KS spinors for multiple bremsstrahlung spin amplitudes, for massive fermions.
- Additional important aims is to handle (Wigner rotations) efficiently and without any approximations, the full spin polarization vectors (= dens. matr.) for any fermion — very important for interfacing MC simulation of production and decay of unstable fermions like τ and top quark.
- With the GPS upgrade of the KS technique we are fully armed to implement completely the above ambitious scenario!

CEEX Amplitudes

Outline of Part II

- Preliminaria: Photon polarization vector,
  Building blocks for bremsstrahlung amplitudes: U and V matrices.
- 1-photon real using KS, isolate IR and non-IR parts, notation,
  and 1-photon virtual, collect old results.
- Warm up: CEEX, zero order $\mathcal{O}(\alpha^0)_{\rm CEEX}$.
- Full scale:, $\mathcal{O}(\alpha^r)_{\rm CEEX}$, $r = 1, 2$.
- Spin structure, connection to decays of final fermion.
- IR cancellations, IR-cut for real photons, etc.
- CPU time considerations and photon spin randomization.
- Appendix on first and second order β's.

CEEX Amplitudes

Photon polarization vector

For a circularly polarized photon with four-momentum k and helicity $\sigma = \pm 1$ we adopt the choice of Kleiss-Stirling and/or Beijing group:

$$(\epsilon^\mu_\sigma(k,\beta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu u_\sigma(\beta)}{\sqrt{2}\;\bar{u}_{-\sigma}(k)u_\sigma(\beta)}, \quad (\epsilon^\mu_\sigma(k,\zeta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu \mathfrak{u}_\sigma(\zeta)}{\sqrt{2}\;\bar{u}_{-\sigma}(k)\mathfrak{u}_\sigma(\zeta)},$$

where β is an arbitrary light-like four-vector $\beta^2 = 0$ (axial gauge).

The second choice with $\mathfrak{u}_\sigma(k,\zeta)$ seem to be ours (not exploited by KS).

Using the Chisholm identity:

$$\bar{u}_\sigma(k)\gamma_\mu u_\sigma(\beta)\;\gamma^\mu = 2u_\sigma(\beta)\;\bar{u}_\sigma(k) + 2u_{-\sigma}(k)\;\bar{u}_{-\sigma}(\beta),$$
$$\bar{u}_\sigma(k)\gamma_\mu \mathfrak{u}_\sigma(\zeta)\;\gamma^\mu = 2\mathfrak{u}_\sigma(\zeta)\;\bar{u}_\sigma(k) - 2u_{-\sigma}(k)\;\bar{\mathfrak{u}}_{-\sigma}(\zeta)$$

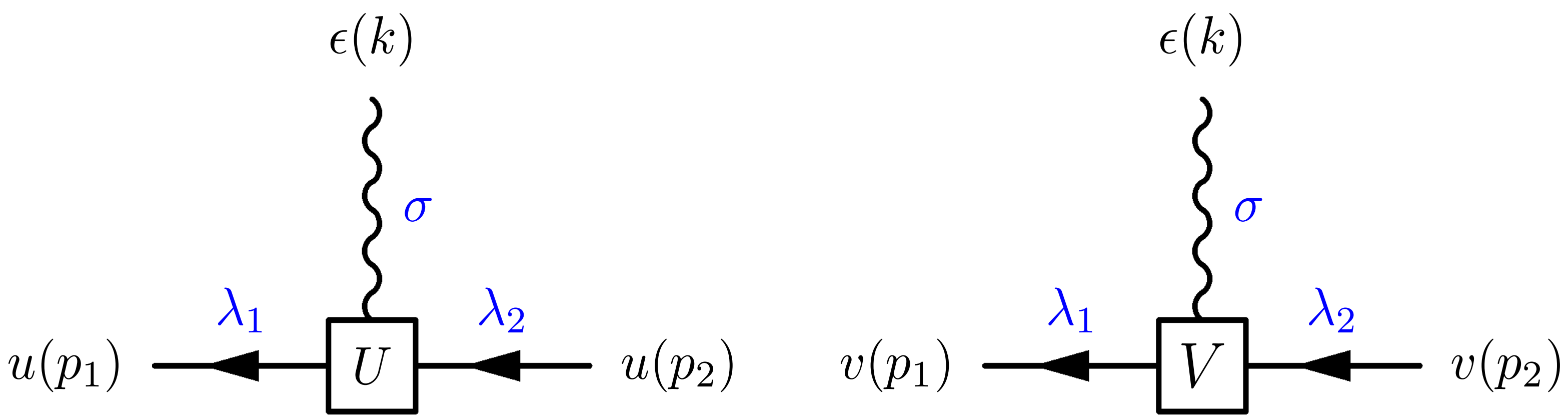
we get useful equivalent expressions:

$$(\not{\epsilon}_\sigma(k,\beta))^* = \frac{\sqrt{2}\,[u_\sigma(\beta)\bar{u}_\sigma(k) + u_{-\sigma}(k)\bar{u}_{-\sigma}(\beta)]}{\bar{u}_{-\sigma}(k)u_\sigma(\beta)},$$
$$(\not{\epsilon}_\sigma(k,\zeta))^* = \frac{\sqrt{2}\,[\mathfrak{u}_\sigma(\zeta)\bar{u}_\sigma(k) - u_{-\sigma}(k)\bar{\mathfrak{u}}_{-\sigma}(\zeta)]}{\sqrt{2\zeta k}}.$$

CEEX Amplitudes

Building blocks for bremsstrahlung amplitudes: U and V matrices.

Basic building blocks in our calculation of the bremsstrahlung amplitudes are:



$$U\begin{bmatrix} p_1 & k & p_2 \\ \lambda_1 & \sigma & \lambda_2 \end{bmatrix} = U^{\sigma}_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = \bar{u}(p_1,\lambda_1)\ \not{\epsilon}^{\star}_{\sigma}(k)\ u(p_2,\lambda_2),$$

$$V\begin{bmatrix} p_1 & k & p_2 \\ \lambda_1 & \sigma & \lambda_2 \end{bmatrix} = V^{\sigma}_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = \bar{v}(p_1,\lambda_1)\ \not{\epsilon}^{\star}_{\sigma}(k)\ v(p_2,\lambda_2).$$

The four-momentum conservation is not assumed in the above vertex-like objects.

CEEX Amplitudes

U and V transition matrices, programmable formulas

In the case of $\epsilon_\sigma(k,\zeta)$ the transition matrices are rather simple:

$$U^+(k,p_1,m_1,p_2,m_2) = \sqrt{2}\begin{bmatrix} \sqrt{\frac{2\zeta p_2}{2\zeta k}} s_+(k,\hat{p_1}), & 0 \\ m_2\sqrt{\frac{2\zeta p_1}{2\zeta p_2}} - m_1\sqrt{\frac{2\zeta p_2}{2\zeta p_1}}, & \sqrt{\frac{2\zeta p_1}{2\zeta k}} s_+(k,\hat{p_2}) \end{bmatrix},$$

$$U^-_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = \left[-U^+_{\lambda_2,\lambda_1}(k,p_2,m_2,p_1,m_1)\right]^*,$$

$$V^\sigma_{\lambda_1,\lambda_2}(k,p_1,m_1,p_2,m_2) = U^\sigma_{-\lambda_1,-\lambda_2}(k,p_1,-m_1,p_2,-m_2).$$

The above expressions is optimized for fast numerical evaluation.

The following general case, with $\epsilon_\sigma(k,\beta)$, looks a little bit more complicated:

$$U^+(k,p_1,m_1,p_2,m_2) = \frac{\sqrt{2}}{s_-(k,\beta)} \times$$

$$\begin{bmatrix} s_+(\hat{p}_1,k)s_-(\beta,\hat{p}_2) + m_1 m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_1}\frac{2\zeta k}{2\zeta p_2}}, & m_1\sqrt{\frac{2\zeta\beta}{2\zeta p_1}} s_+(k,\hat{p}_2) + m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_2}} s_+(\hat{p}_1,k) \\ m_1\sqrt{\frac{2\zeta k}{2\zeta p_1}} s_-(\beta,\hat{p}_2) + m_2\sqrt{\frac{2\zeta k}{2\zeta p_2}} s_-(\hat{p}_1,\beta), & s_-(\hat{p}_1,\beta)s_+(k,\hat{p}_2) + m_1 m_2\sqrt{\frac{2\zeta\beta}{2\zeta p_1}\frac{2\zeta k}{2\zeta p_2}} \end{bmatrix}$$

NB. They are not M-matrices of KS, but rather products of them.

CEEX Amplitudes

Diagonality property of U and V

For $p_1 = p_2 = p$ the matrices U and V become diagonal:

$$U \begin{bmatrix} p & k & p \\ \lambda_1 & \sigma & \lambda_2 \end{bmatrix} = V \begin{bmatrix} p & k & p \\ \lambda_1 & \sigma & \lambda_2 \end{bmatrix} = b_\sigma(k,p)\ \delta_{\lambda_1 \lambda_2},$$

$$b_\sigma(k,p) = \sqrt{2}\, \frac{\bar{u}_\sigma(k)\ \not{p}\ \mathfrak{u}_\sigma(\zeta)}{\bar{u}_{-\sigma}(k)\mathfrak{u}_\sigma(\zeta)} = \sqrt{2}\, \sqrt{\frac{2\zeta p}{2\zeta k}} s_\sigma(k,\hat{p}).$$

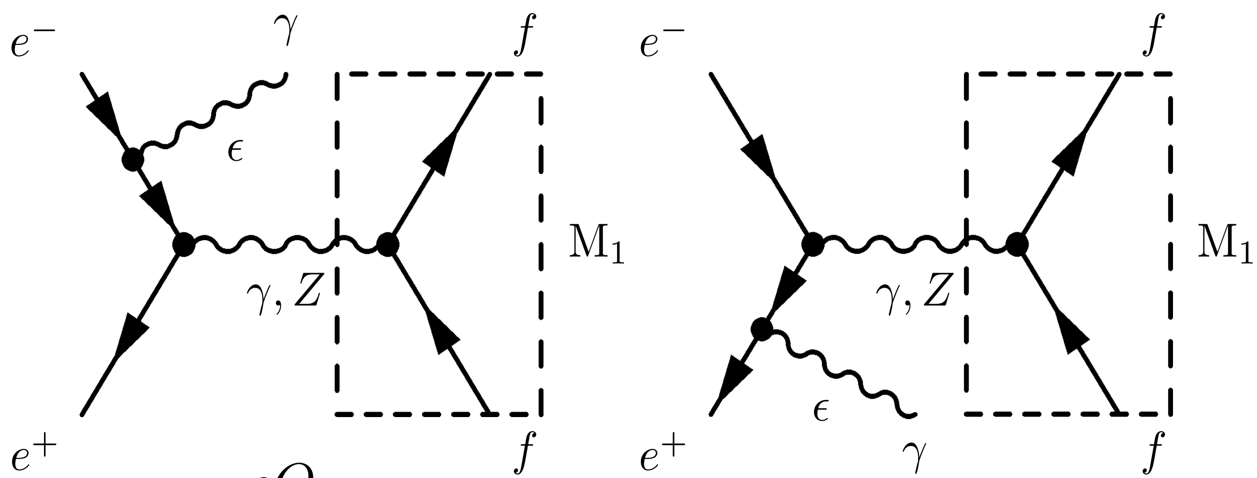
In a more general case of $\epsilon(k,\beta)$ diagonality also holds:

$$b_\sigma(k,p) = \frac{\sqrt{2}}{s_{-\sigma}(k,\beta)} \left(s_{-\sigma}(\beta,\hat{p}) s_\sigma(\hat{p},k) + \frac{m^2}{2\zeta\hat{p}} \sqrt{(2\beta\zeta)\ (2\zeta k)} \right)$$

Thanks to the above diagonality we easily obtain/explore the soft limit of the multi-photon amplitudes.

CEEX Amplitudes

ISR First order real 1-photon

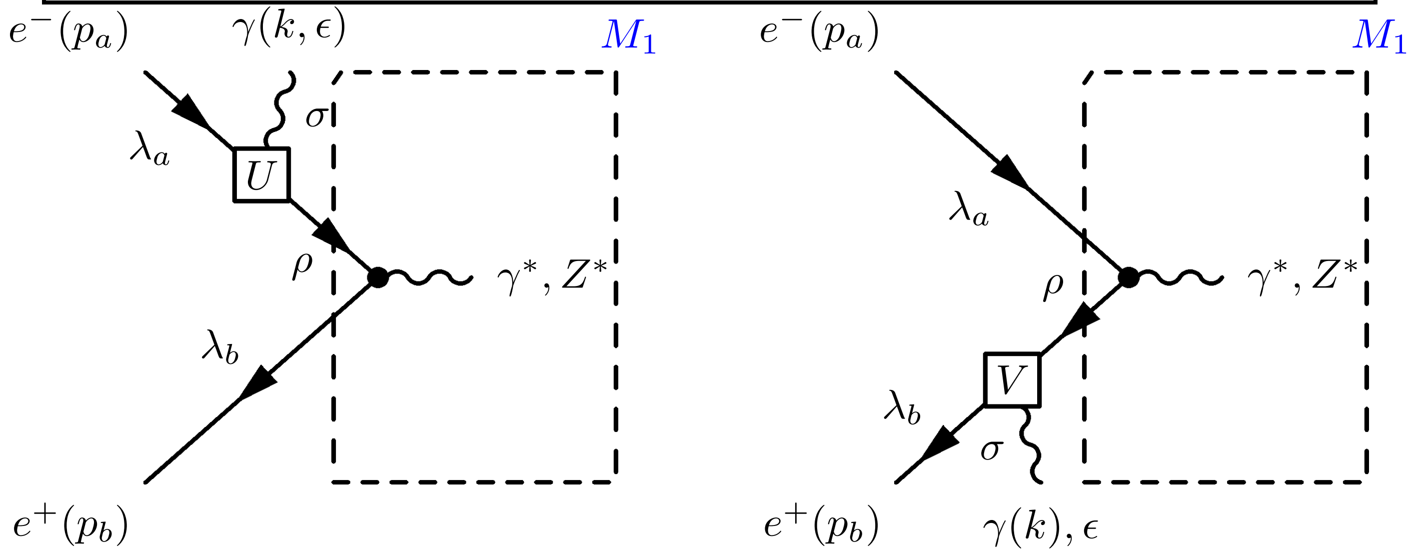


$$\mathcal{M}_1^{\rm ISR}\begin{pmatrix} p_a\, p_b\, k_1 \\ \lambda_a\, \lambda_b\, \sigma_1 \end{pmatrix} = \frac{eQ_e}{2k_1 p_a}\; \bar{v}(p_b,\lambda_b)\, \mathbf{M}_1\, (\not{p}_a + m - \not{k}_1)\; \not{\epsilon}^{\star}_{\sigma_1}(k_1)\; u(p_a,\lambda_a)$$
$$+\frac{eQ_e}{2k_1 p_b}\; \bar{v}(p_b,\lambda_b)\; \not{\epsilon}^{\star}_{\sigma_1}(k_1)\; (-\not{p}_b + m + \not{k}_1)\, \mathbf{M}_1\; u(p_a,\lambda_a)$$

- Separate parts containing $\not{p} \pm m$ and $\not{k}_1$, (IR and finite)
- Apply completeness relation for $\not{p} \pm m$ and $\not{k}_1$,
- Express bispinor components in terms of U and V matrices.
- Use diagonality property.

CEEX Amplitudes

ISR in terms of U and V matrices; separation of IR



Using compact notation $\begin{bmatrix} p \\ \lambda \end{bmatrix} \equiv \begin{bmatrix} p_b & p_a & p_c & p_d \\ \lambda_b & \lambda_a & \lambda_c & \lambda_d \end{bmatrix} \equiv \begin{bmatrix} p_b & p_a \\ \lambda_b & \lambda_a \end{bmatrix}_{[cd]}$ we get:

$$\mathfrak{M}_1^{\rm ISR}\begin{pmatrix} p_a & p_b & p_a & p_d & k_1 \\ \lambda_a & \lambda_b & \lambda_c & \lambda_b & \sigma_1 \end{pmatrix} \equiv \mathfrak{M}_1^{\rm ISR}\begin{pmatrix} p & k_1 \\ \lambda & \sigma_1 \end{pmatrix} =$$

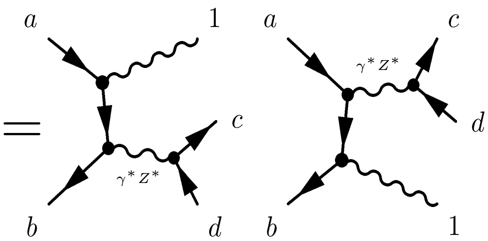
$$= \frac{eQ_e}{2k_1p_a}\sum_\rho \mathfrak{B}\begin{bmatrix} p_b & p_a \\ \lambda_b & \rho \end{bmatrix}_{[cd]} U\begin{bmatrix} p_a & k_1 & p_a \\ \rho & \sigma_1 & \lambda_a \end{bmatrix} - \frac{eQ_e}{2k_1p_b}\sum_\rho V\begin{bmatrix} p_b & k_1 & p_b \\ \lambda_b & \sigma_1 & \rho \end{bmatrix}\mathfrak{B}\begin{bmatrix} p_b & p_a \\ \rho & \lambda_a \end{bmatrix}_{[cd]}$$

$$- \frac{eQ_e}{2k_1p_a}\sum_\rho \mathfrak{B}\begin{bmatrix} p_b & k_1 \\ \lambda_b & \rho \end{bmatrix}_{[cd]} U\begin{bmatrix} k_1 & k_1 & p_a \\ \rho & \sigma_1 & \lambda_a \end{bmatrix} + \frac{eQ_e}{2k_1p_b}\sum_\rho V\begin{bmatrix} p_b & k_1 & k_1 \\ \lambda_b & \sigma_1 & \rho \end{bmatrix}\mathfrak{B}\begin{bmatrix} k_1 & p_a \\ \rho & \lambda_a \end{bmatrix}_{[cd]}$$

Now we may exploit **diagonality** of U and V in IR parts, see next slide $\longrightarrow$

Real 1-γ ISR summary

IR and non-IR clearly separated:

$$\mathfrak{M}\binom{p\,k_1}{\lambda\,\sigma_1} =$$ 

$$= \mathfrak{s}^{\{1\}}_{[1]} \mathfrak{B}\begin{bmatrix} p \\ \lambda \end{bmatrix} + r^{\{1\}}\binom{p\,k_1}{\lambda\,\sigma_1}$$

$$\mathfrak{s}^{\{1\}}_{[1]} \equiv \mathfrak{s}^{\{1\}}_{\sigma_1}(k_1) \equiv \mathfrak{s}^{(a)}_{[1]} + \mathfrak{s}^{(b)}_{[1]},$$

$$\mathfrak{s}^{(a)}_{[1]} = eQ_e\,\frac{b_\sigma(k,p_a)}{-2k_1p_a},\quad \mathfrak{s}^{(b)}_{[1]} = eQ_e\,\frac{b_\sigma(k,p_b)}{2k_1p_b},$$

$$r^{\{1\}}\binom{p\,k_1}{\lambda\,\sigma_1} = -\frac{eQ_e}{2k_1p_a}\sum_{\sigma_1'} \mathfrak{B}\begin{bmatrix} p_b & k_1 \\ \lambda_b & \sigma_1' \end{bmatrix}_{[cd]} U\begin{bmatrix} k_1 & k_1 & p_a \\ \sigma_1' & \sigma_1 & \lambda_a \end{bmatrix}$$
$$+\frac{eQ_e}{2k_1p_b}\sum_{\sigma_1'} V\begin{bmatrix} p_b & k_1 & k_1 \\ \lambda_b & \sigma_1 & \sigma_1' \end{bmatrix} \mathfrak{B}\begin{bmatrix} k & p_a \\ \sigma_1' & \lambda_a \end{bmatrix}_{[cd]}$$

Introduce super-compact notation:

$$(if) = 2k_i\cdot p_f,\ f = a,b,c,d,$$

$$\begin{bmatrix} p \\ \lambda \end{bmatrix} \equiv \begin{bmatrix} p_b & p_a & p_c & p_d \\ \lambda_b & \lambda_a & \lambda_c & \lambda_d \end{bmatrix} \equiv \begin{bmatrix} p_b & p_a \\ \lambda_b & \lambda_a \end{bmatrix}_{[cd]} \equiv {}_{[ba]}\begin{bmatrix} p_c & p_d \\ \lambda_c & \lambda_d \end{bmatrix} \equiv {}_{[bacd]}$$

and summation understood over **primed** (spin) indices.

An get super-compact expression:

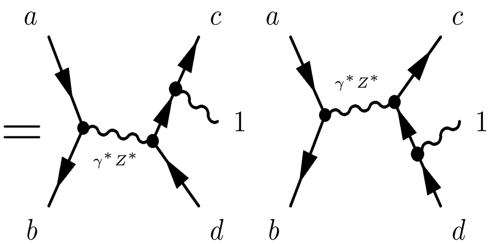
$$r^{\{1\}}\binom{p\,k_1}{\lambda\,\sigma_1} = \frac{eQ_e}{-(1a)}\mathfrak{B}_{[b1'][cd]}U_{[1'1a]} + \frac{eQ_e}{(1b)}V_{[b11']}\mathfrak{B}_{[1'a][cd]}\Big\}$$

This notation will be useful for 2 and more photons.

Real 1-γ FSR summary

IR and non-IR clearly separated:

$$\mathfrak{M}\binom{p\, k_1}{\lambda\, \sigma_1} =$$



$$= \mathfrak{s}^{\{0\}}_{[1]} \mathfrak{B}\begin{bmatrix} p \\ \lambda \end{bmatrix} + r^{\{0\}}\binom{p\, k_1}{\lambda\, \sigma_1}$$

Using super-compact notation:

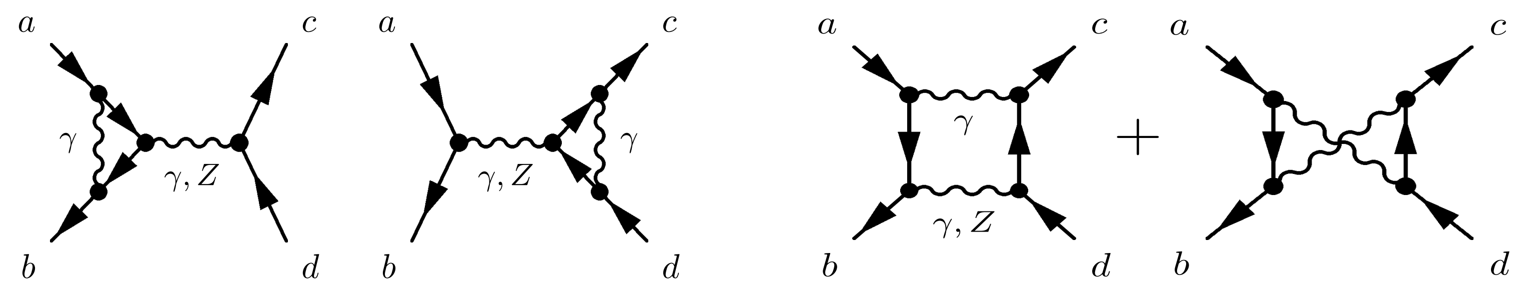
$$\mathfrak{s}^{\{0\}}_{[1]} \equiv \mathfrak{s}^{\{0\}}_{\sigma_1}(k_1) \equiv \mathfrak{s}^{(c)}_{[1]} + \mathfrak{s}^{(d)}_{[1]},$$

$$\mathfrak{s}^{(c)}_{[1]} = eQ_f\, \frac{b_\sigma(k, p_c)}{2k_1 p_c}, \quad \mathfrak{s}^{(d)}_{[1]} = eQ_f\, \frac{b_\sigma(k, p_d)}{-2k_1 p_d},$$

$$r^{\{0\}}\binom{p\, k_1}{\lambda\, \sigma_1} = \frac{eQ_f}{(1c)} U_{[c11']} \mathfrak{B}_{[ba][1'd]} + \frac{eQ_f}{-(1d)} \mathfrak{B}_{[ba][c1']} V_{[1'1d]} \}$$

CEEX Amplitudes

First order, one virtual photon



$$\mathcal{M}^{(1)}_0\left({p \atop \lambda};X\right) = \mathfrak{B}\left({p \atop \lambda};X\right)\left[1 + Q_e^2 F_1(s,m_\gamma) + Q_f^2 F_1(s,m_\gamma)\right] + \mathcal{M}_{\rm box}\left({p \atop \lambda};X\right),$$

F_1 is the electric formfactor regularized with m_γ, keep exact final fermion mass.

We omit F_2, this is justified for light final fermions. To be restored later on.

Spin amplitudes for γ-γ and γ-Z boxes are:

$$\mathcal{M}_{\rm Box}\left({p \atop \lambda};X\right) =$$

$$= ie^2 \sum_{B=\gamma,Z} \frac{g^{e,B}_{\lambda_a} g^{f,B}_{-\lambda_a}\; T_{\lambda_c\lambda_a} T'_{\lambda_b\lambda_d} + g^{e,B}_{\lambda_a} g^{f,B}_{\lambda_a}\; U'_{\lambda_c\lambda_b} U_{\lambda_a\lambda_d}}{X^2 - M_B{}^2 + i\Gamma_B X^2/M_B}\; \delta_{\lambda_a,-\lambda_b}\delta_{\lambda_c,-\lambda_d}$$

$$\times\, \frac{\alpha}{\pi} Q_e Q_f \left[\delta_{\lambda_a,\lambda_c}\; f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s,t,u) - \delta_{\lambda_a,-\lambda_c}\; f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s,u,t)\right],$$

where $\bar{M}_Z^2 = M_Z^2 - iM_Z\Gamma_Z$, $\bar{M}_\gamma^2 = m_\gamma^2$. and function $f_{\rm BDP}$ is defined in Brown, Decker and Paschos (1984); Mandelstam variables s, t and u defined as usual.

CEEX Amplitudes

Warm up example: resonant CEEX, zero-th order $\mathcal{O}(\alpha^0)_{\rm CEEX}$

$$\sigma^{(0)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_{n+2}(p_c, p_d, k_1, \dots, k_n)\ \frac{1}{4} \sum_{\lambda, \sigma_j} \left| \mathfrak{M}_n^{(r)} \begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \right|^2$$

$$\mathfrak{M}_n^{(0)} \begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} = e^{\alpha B_4^\star(p_a,\dots,p_d)} \sum_{\wp \in \mathcal{P}} \frac{X_\wp^2}{s_{cd}}\ \mathfrak{B}\left({p \atop \lambda}; X_\wp \right) \prod_{i=1}^{n} \mathfrak{s}_{[i]}^{\{\wp_i\}}$$

Notation: $\binom{p}{\lambda} \equiv \begin{pmatrix} p_a & p_b & p_c & p_d \\ \lambda_a & \lambda_b & \lambda_c & \lambda_d \end{pmatrix}$, $s_{cd} = (p_c + p_d)^2$, $d\tau_n \equiv$ Lorentz inv. phase space.

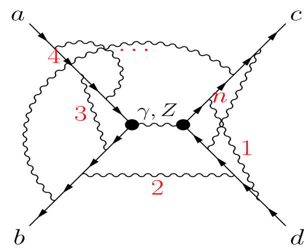
- The *coherent sum* is taken over set of 2^n partitions: $\mathcal{P} = \{\ (0,0,0,\dots,0), (1,0,0,\dots,0), (0,1,0,\dots,0), (1,1,0,\dots,0), \dots\ (1,1,1,\dots,1)\}$.
- In single partition $\wp = (\wp_1, \dots, \wp_i, \dots, \wp_n)$, $\wp_i = 1, 0$ for ISR, FSR so that $X_\wp = p_a + p_b - \sum_{i=1}^{n} \wp_i k_i \equiv$ the 4-mom. in **resonance** propagator.
- $\mathfrak{s}_{[i]}^{\{\omega\}} \equiv j_{\{\omega\}}(k_i) \cdot \epsilon_{\sigma_i}$ are IR-div. real-photon soft-factors with ISR/FSR currents: $j^\mu_{\{1\}}(k_i) = -\frac{eQ_e}{2}\left(\frac{p_a^\mu}{k_i p_a} - \frac{p_b^\mu}{k_i p_b} \right)$, $j^\mu_{\{0\}}(k_i) = \frac{eQ_e}{2}\left(\frac{p_c^\mu}{k_i p_c} - \frac{p_d^\mu}{k_i p_d} \right)$, and ϵ_{σ_i} are our KS/Beijing photon polarization vectors (axial gauge).
- $B_4^\star(p_a, \dots, p_d)$ is IR virtual form-factor with the **resonance** part, see next slide.

CEEX Amplitudes

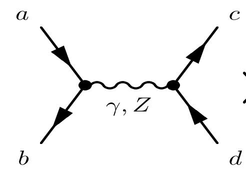
Virtual Formfactor

Factorization of virtual IR by Yennie-Frautschi-Suura (1961):

$$\sum_{n=0}^{\infty}$$



$$= e^{\alpha B_4}$$



$$\times (1 + \Delta_{\rm fin.})$$

where $B_4(p_a, ..., p_d) = \int \frac{d^4 k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} \left| J_I(k) - J_F(k) \right|^2,$

$$J_I = eQ_e\big(\hat{J}_a(k) - \hat{J}_b(k)\big), \quad J_F = eQ_f\big(\hat{J}_c(k) - \hat{J}_d(k)\big), \quad \hat{J}_f^{\mu}(k) \equiv \frac{2p_f^{\mu} + k^{\mu}}{k^2 + 2kp_f + i\epsilon}$$

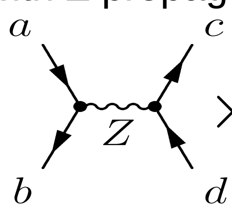
Modification for Z resonance by Greco, Pancheri and Srivastava (1975,1980):

$\boxed{B_4 \to B_4^{\star}}$: [Notation: $M^2 = M_Z^2 - iM\Gamma$]

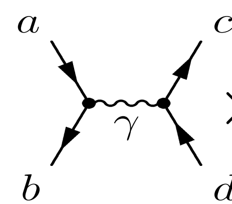
$$|J_I - J_F|^2 \to |J_I(k)|^2 + |J_F(k)|^2 - 2\Re\big(J_I(k) \cdot J_F^*(k)\big) \frac{(p_a + p_b)^2 + M^2}{(p_a + p_b - k)^2 + M^2}.$$

Only diagrams with Z propagator are concerned:

$$e^{\alpha B_4^{\star}}$$



$$\times (1 + \Delta_{\rm fin.}) + e^{\alpha B_4}$$



$$\times (1 + \Delta'_{\rm fin.})$$

CEEX Amplitudes

Limit (1): no resonances at all

$$\sum_{\wp\in\mathcal{P}} e^{\alpha B_4^\star(X_\wp)} \frac{X_\wp{}^2}{s_{cd}} \mathfrak{B}\binom{p}{\lambda}; X_\wp) \prod_{i=1}^{n} \mathfrak{s}_{[i]}^{\{\wp_i\}} \Longrightarrow e^{\alpha B_4^\star} \mathfrak{B}\binom{p}{\lambda}; P) \prod_{i=1}^{n} \left(\mathfrak{s}_{[i]}^{\{0\}} + \mathfrak{s}_{[i]}^{\{1\}}\right),$$

because $\sum_{\wp\in\mathcal{P}} \prod_{i=1}^{n} \mathfrak{s}_{[i]}^{\{\wp_i\}} \equiv \prod_{i=1}^{n} \left(\mathfrak{s}_{[i]}^{\{0\}} + \mathfrak{s}_{[i]}^{\{1\}}\right).$ [$P = p_a + p_b$, **for example.**]

Nevertheless it is good to keep sum over partition even for the non-resonant case! (Better LL summations).

CEEX for non-resonant is also very valuable! (The only hope for Bhabha!)

Limit (2): very narrow resonance

$$|\mathfrak{M}_n^{(0)}|^2 = \sum_{\wp\in\mathcal{P}} \sum_{\wp'\in\mathcal{P}} e^{\alpha B_4^\star(X_\wp)} e^{\alpha(B_4^\star(X_{\wp'}))^*} \mathfrak{B}\binom{p}{\lambda}; X_\wp) \mathfrak{B}\binom{p}{\lambda}; X_{\wp'})^* \prod_{i=1}^{n} \mathfrak{s}_{[i]}^{\{\wp_i\}} \prod_{j=1}^{n} \mathfrak{s}_{[j]}^{\{\wp'_i\}*}$$

$$\Longrightarrow e^{2\alpha\Re B_2(p_a,p_b)} e^{2\alpha\Re B_2(p_c,p_d)} \sum_{\wp\in\mathcal{P}} \left|\mathfrak{B}\binom{p}{\lambda}; X_\wp)\right|^2 \prod_{i=1}^{n} \left|\mathfrak{s}_{[i]}^{\{\wp_i\}}\right|^2.$$

Neglect of ISR*FSR interferences implies that terms $\wp \neq \wp'$ **drop out.**

This approximation as used in KORALZ/YFS3 for LEP1.

At LEP2 it cannot be justified any more.

CEEX Amplitudes

Full Scale CEEX $\mathcal{O}(\alpha^r)$ master formula

Polarized total x-section:

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_n(p_a+p_b; p_c, p_d, k_1, \dots, k_n)\; e^{2\alpha \Re B_4} \sum_{\sigma_i,\lambda,\bar\lambda} \sum_{i,j,l,m=0}^{3} \hat\varepsilon_a^i \hat\varepsilon_b^j \sigma^i_{\lambda_a\bar\lambda_a} \sigma^j_{\lambda_b\bar\lambda_b} \mathfrak{M}_n^{(r)} \begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \left[\mathfrak{M}_n^{(r)} \begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \bar\lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \right]^\star \sigma^l_{\bar\lambda_c\lambda_c} \sigma^m_{\bar\lambda_d\lambda_d} \hat h_c^l \hat h_c^m$$

CEEX amplitudes:

$$\mathfrak{M}_n^{(1)} \begin{pmatrix} p & k_1 & \dots & k_n \\ \lambda & \sigma_1 & \dots & \sigma_n \end{pmatrix} = \sum_{\wp\in\mathcal{P}} \prod_{i=1}^{n} \mathfrak{s}_{[i]}^{\{\wp_i\}} \left\{ \beta_0^{(1)} \begin{pmatrix} p \\ \lambda \end{pmatrix}; X_\wp \right) + \sum_{j=1}^{n} \frac{\beta^{(1)}_{1\{\wp_j\}} \left(\begin{smallmatrix} p & k_j \\ \lambda & \sigma_j \end{smallmatrix}; X_\wp \right)}{\mathfrak{s}_{[j]}^{\{\wp_j\}}} \right\}$$

$$\mathfrak{M}_n^{(2)} \begin{pmatrix} p & k_1 & \dots & k_n \\ \lambda & \sigma_1 & \dots & \sigma_n \end{pmatrix} = \sum_{\wp\in\mathcal{P}} \prod_{i=1}^{n} \mathfrak{s}_{[i]}^{\{\wp_i\}}$$

$$\times \left\{ \beta_0^{(2)} \left(\begin{smallmatrix} p \\ \lambda \end{smallmatrix}; X_\wp \right) + \sum_{j=1}^{n} \frac{\beta^{(1)}_{2\{\wp_j\}} \left(\begin{smallmatrix} p & k_j \\ \lambda & \sigma_j \end{smallmatrix}; X_\wp \right)}{\mathfrak{s}_{[j]}^{\{\wp_j\}}} + \sum_{1\le j<l\le n} \frac{\beta^{(2)}_{2\{\wp_j,\wp_l\}} \left(\begin{smallmatrix} p & k_j & k_l \\ \lambda & \sigma_j & \sigma_l \end{smallmatrix}; X_\wp \right)}{\mathfrak{s}_{[j]}^{\{\wp_j\}} \mathfrak{s}_{[l]}^{\{\wp_l\}}} \right\}$$

For details see next slides.

CEEX Amplitudes

SPIN STRUCTURE

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\tau_n(p_a+p_b;p_c,p_d,k_1,\dots,k_n)\; e^{2\Re B_4} \sum_{\sigma_i,\lambda,\bar\lambda}\; \sum_{i,j,l,m=0}^{3}$$
$$\hat\varepsilon_a^i \hat\varepsilon_b^j \sigma^i_{\lambda_a\bar\lambda_a} \sigma^j_{\lambda_b\bar\lambda_b} \mathfrak{M}_n^{(r)}\begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \left[\mathfrak{M}_n^{(r)}\begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \bar\lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix}\right]^{\star} \sigma^l_{\bar\lambda_c\lambda_c} \sigma^m_{\bar\lambda_d\lambda_d} \hat h_c^l \hat h_c^m$$

- σ^k for $k = 1,2,3$ are Pauli matrices and $\sigma^0_{\lambda,\mu} = \delta_{\lambda,\mu}$ is unit matrix.
- $\hat\varepsilon_1^a, \hat\varepsilon_2^b, a,b = 1,2,3$ are the components of the conventional spin polarization vectors of the **beam** $e^{\pm}$ and $\hat\varepsilon_i^0 \equiv 1$ in $e^{\pm}$ rest frame [$\hat\varepsilon_i \cdot p_i = m_e$].
- $\hat h_3^c \hat h_4^d$ are polarimeter vectors of outgoing fermions. They carry spin information to **final fermion decay** processes [Notation: $\hat h_i \cdot p_i = m_f$].
- $\hat\varepsilon_i^a$ and $\hat h_i^a$ are **primarily** defi ned in the so called GPS frames of the corresponding fermions. (Important for the use of Pauli matrices!)

CEEX Amplitudes

$\mathcal{O}(\alpha^1)_{\rm CEEX}$ Beta's

$$\mathfrak{M}^{(1)}_n \left({p\,k_1 \atop \lambda\,\sigma_1} \dots {k_n \atop \sigma_n}\right) = \sum_{\wp\in\mathcal{P}} \prod_{i=1}^{n} \mathfrak{s}^{\{\wp_i\}}_{[i]} \left\{ \beta^{(1)}_0 \left({p \atop \lambda}; X_\wp\right) + \sum_{j=1}^{n} \frac{\beta^{(1)}_{1\{\wp_j\}}\left({p\,k_j \atop \lambda\,\sigma_j}; X_\wp\right)}{\mathfrak{s}^{\{\wp_j\}}_{[j]}} \right\}$$

$$\beta^{(1)}_0\left({p \atop \lambda}; X\right) = \mathfrak{B}\left({p \atop \lambda}; X\right)\left(1 + \delta^{(1)}_{Virt}\right) + \mathcal{R}_{\rm Box}\left({p \atop \lambda}; X\right)$$

$$\beta^{(1)}_{1\{1\}}\left({p\,k_j \atop \lambda\,\sigma_j}; X\right) = r^{\{1\}}\left({p\,k_j \atop \lambda\,\sigma_j}; X\right)$$

$$\beta^{(1)}_{1\{0\}}\left({p\,k_j \atop \lambda\,\sigma_j}; X\right) = r^{\{0\}}\left({p\,k_j \atop \lambda\,\sigma_j}; X\right) + \left(\frac{(p_c+p_d+k_j)^2}{(p_c+p_d)^2} - 1\right)\mathfrak{B}\left({p \atop \lambda}; X\right)$$

The most important: β's are IR-finite!

Formal defi nition (omitting inessential arguments/indices) of β's are :

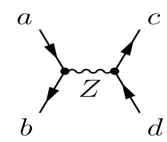
$$\beta^{(1)}_0 = \left(e^{\alpha B_4}\mathcal{M}^{(1)}_0\right)\Big|_{\mathcal{O}(\alpha^1)}$$

$$\beta^{(1)}_{1\{\omega\}}\left({k_j \atop \sigma_j}\right) = \mathcal{M}^{(1)}_{1\{\omega\}}\left({k_j \atop \sigma_j}\right) - \beta^{(1)}_0\ \mathfrak{s}^{\{\omega\}}_{[j]}$$

Complications due to **Z resonance** see next slide.

CEEX Amplitudes

Complication due to resonance

In amplitude part **proportional** to resonant propagator 

we replace:

$$B_4 \Rightarrow B_4^{\star} = B_I(p_a,p_b) + B_F(p_c,p_b) + B^{\star}_{IFI}(p_a,p_b,p_c,p_b,X_{\wp})$$

$$\int d^4k\; |J_I - J_F|^2 \Rightarrow \int d^4k\; |J_I(k)|^2 + |J_F(k)|^2 - 2\Re(J_I(k)\cdot J_F^*(k))\,\frac{X_{\wp})^2+M^2}{(X_{\wp}-k)^2+M^2}.$$

[Notation: $M^2 = M_Z^2 - iM\Gamma$]

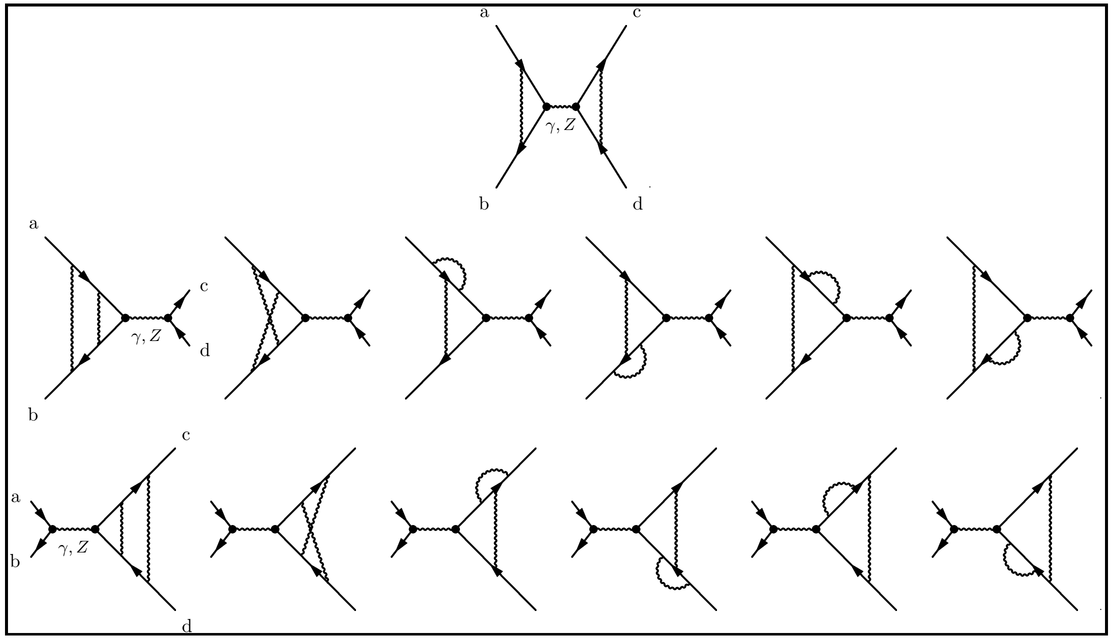
Consequently:

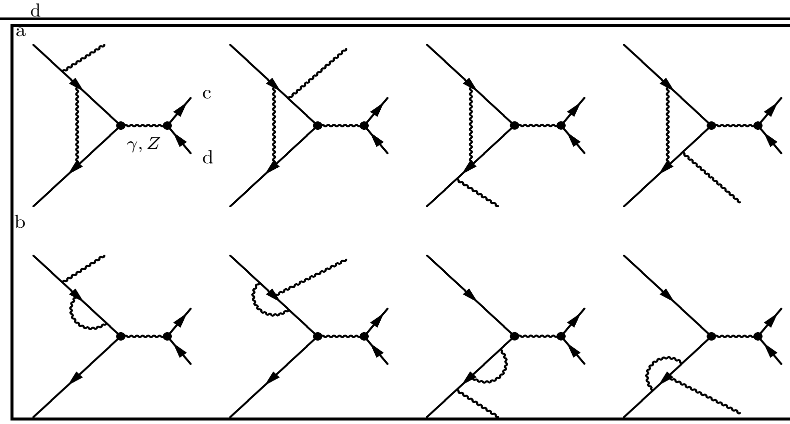
$\beta_0^{(1)} = \left(e^{2\alpha B_4^{\star}}\mathcal{M}_0^{(1)}\right)\Big|_{\mathcal{O}(\alpha^1)}$ induces additional subtraction of order $\ln\frac{\Gamma_Z}{M_Z}$ in $\mathcal{R}_{\mathrm{Box}}$.

The overall virtual formfactor in the master formula gains partition dependence in the resonance part: $e^{2\alpha B_4^{\star}} = e^{2\alpha B_4(p_a,p_b,p_c,p_d)+2\alpha\delta B_4^{\star}(p_a,p_b,p_c,p_d;X_{\wp})}$.

CEEX Amplitudes

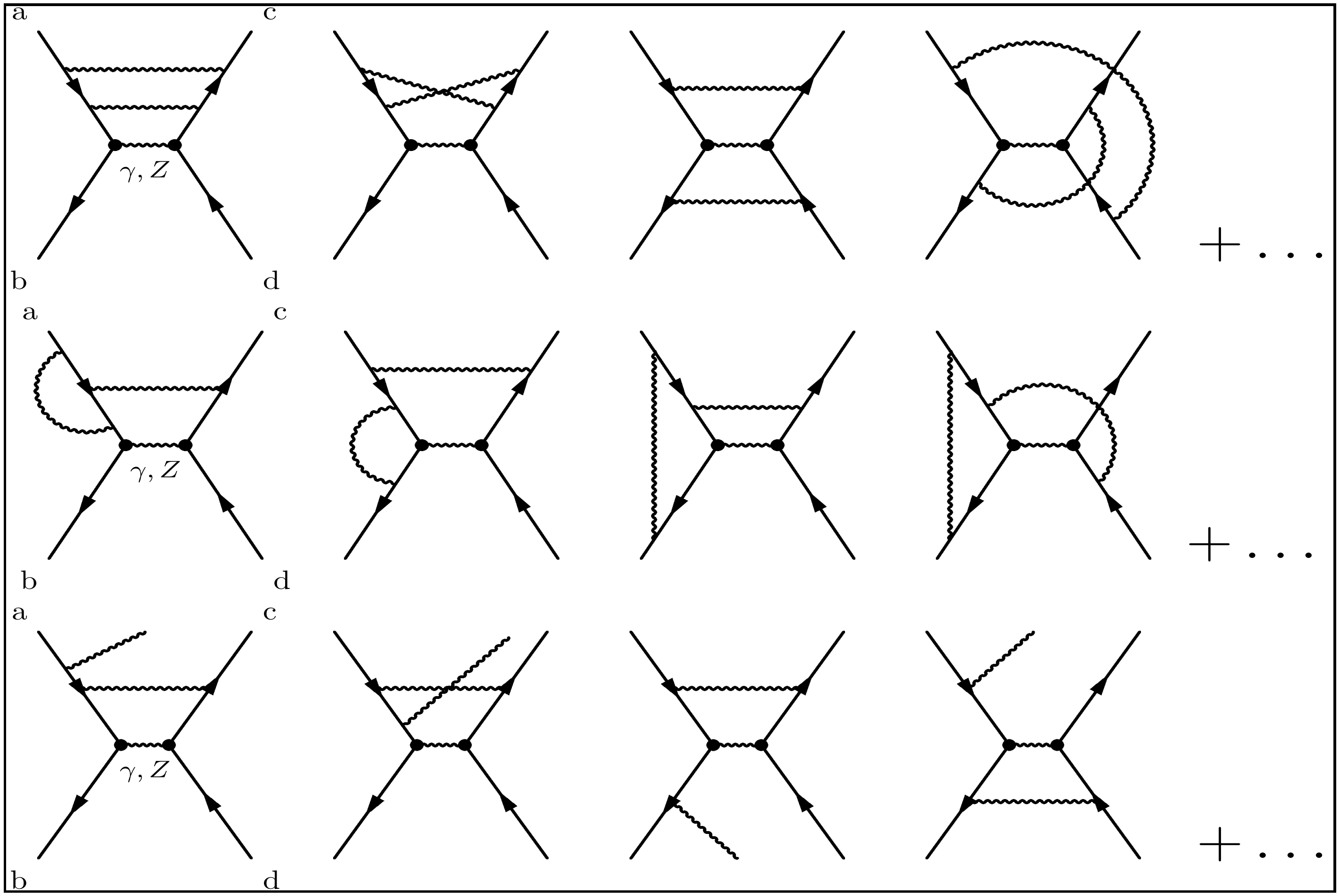
$\mathcal{O}(\alpha^2)_{\rm CEEX}$ **Beta's**





CEEX Amplitudes

$\mathcal{O}(\alpha^2)_{\rm CEEX}$ **Shopping list**



Never ever combine with real emission! (Keep photon mass and publish.)

Never ever square and spin sum! In the M.C. real emission is added anyway.

CEEX Amplitudes

Traditional real photon regulator.

Optionally, the traditional IR-cut $k^0 > \varepsilon\sqrt{s}/2$ on real γ's can be introduced.

Phase space integral $m_\gamma < k^0 < \varepsilon\frac{\sqrt{s}}{2}$ done analiticaly (rigorously) $\Rightarrow e^{2\alpha\tilde{B}_4}$ factor,

where: $\tilde{B}_4(p_a,\dots,p_d) = Q_e^2\tilde{B}_2(p_a,p_b) + Q_f^2\tilde{B}_2(p_c,p_d)$

$+ Q_eQ_f\tilde{B}_2(p_a,p_c) + Q_eQ_f\tilde{B}_2(p_b,p_d) - Q_eQ_f\tilde{B}_2(p_a,p_d) - Q_eQ_f\tilde{B}_2(p_b,p_c)$,

$$\tilde{B}_2(p,q) \equiv \int\limits_{k^0<\varepsilon\sqrt{s}/2} \frac{d^3k}{k^0}\frac{(-1)}{8\pi^2}\left(\frac{p}{kp}-\frac{q}{kq}\right)^2.$$

New Master Formula:

$$\sigma^{(r)} = \sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{k_i^0>\varepsilon\sqrt{s}/2} d\tau_n(p_a+p_b;p_c,p_d,k_1,\dots,k_n)\; e^{2\alpha\Re B_4+2\alpha\tilde{B}_4}\sum_{\sigma_i,\lambda,\bar{\lambda}}\;\sum_{i,j,l,m=0}^{3}$$
$$\hat{\varepsilon}_a^i\hat{\varepsilon}_b^j\sigma^i_{\lambda_a\bar{\lambda}_a}\sigma^j_{\lambda_b\bar{\lambda}_b}\mathfrak{M}_n^{(r)}\begin{pmatrix} p\, k_1\, k_2 \dots k_n \\ \lambda\,\sigma_1\,\sigma_2 \dots \sigma_n\end{pmatrix}\left[\mathfrak{M}_n^{(r)}\begin{pmatrix} p\, k_1\, k_2 \dots k_n \\ \bar{\lambda}\,\sigma_1\,\sigma_2 \dots \sigma_n\end{pmatrix}\right]^\star \sigma^l_{\bar{\lambda}_c\lambda_c}\sigma^m_{\bar{\lambda}_d\lambda_d}\hat{h}_c^l\hat{h}_c^m$$

The resonant part (if present) is multiplied by $e^{2\alpha\delta B_4^\star(p_a,p_b,p_c,p_d;X_\wp)}$

CEEX Amplitudes

IR-cancellations

- IR-cancellations in case of traditional energy-cut regulator is manifest.
- In original master formula with m_γ regulator one may check IR-fi niteness with by **analytical** partial differentiation with respect to the photon mass: $\partial\sigma^{(r)}/\partial m_\gamma = 0$. (G. Burgers (1989) unpublished).
- The classical method of YFS (1961) relies on the techniques of the Melin transform.

NB. Version with m_γ regulator is perfectly implementable in the M.C.

CEEX Amplitudes

CPU considerations: photon spin randomization

The single spin amplitude $\mathfrak{M}_n^{(1)}$ contains already $2^n(n+1)$ terms (2^n due to ISR/FSR partitions). The grand sum over spins counts $2^n 4^4 4^4 = 2^{n+16}$ terms!!! Altogether we expect up to $N \sim n 2^{2n+16}$ operations in the CPU time expensive complex (16bytes) arithmetics. Typically in $e^- e^+ \to \mu^- \mu^+$ the average photon multiplicity with $k^0 > 1MeV$ is about 3, corresponding to $N \sim 10^7$ terms. In a sample of 10^4 MC events there will be a couple events with $n = 10$ and $N = 10^{12}$ terms. Partial solutions: $\sum_a \varepsilon_i^a \sigma_{\lambda\bar{\lambda}}^a$ and the $\mathfrak{s}$-factors evaluated only once, stored and reused (save 2^8).

The trick of **photon spin randomization** speeds up substantially the numerical calculation in the Monte Carlo program: Instead of evaluating the sum over photon spins $\sigma_i, i = 1, ..., n$ we generate randomly one spin sequence of $(\sigma_1, ..., \sigma_n)$ per MC event and the MC weight is calculated only for this particular spin sequence! We save one hefty 2^n factor in the CPU time!

The formal proof of the correctness of this method can be found in Sect. 4 of **"Guide to practical Monte Carlo methods"**, (1998), `http://wwwcn.cern/~jadach`.

APPENDIX A: More on $\mathcal{O}(\alpha^1)_{\rm CEEX}$ virtual's

The IR-finite $\delta^{(1)}_{Virt}$ reads explicitly:

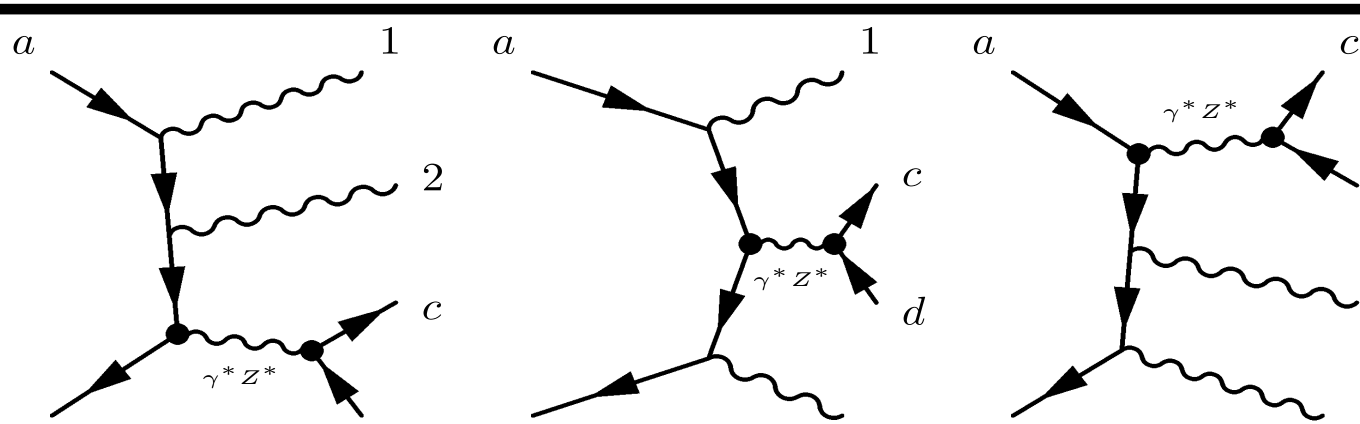
$$\delta^{(1)}_{Virt}(s) = Q_e^2 F_1(s, m_\gamma) + Q_f^2 F_1(s, m_\gamma) - Q_e^2 \alpha B_2(s, m_\gamma) - Q_f^2 \alpha B_2(s, m_\gamma).$$

The $\mathfrak{R}_{\rm Box}$ is effectively obtained from $\mathcal{M}_{\rm Box}$ by:

$f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s, t, u) \Rightarrow f_{\rm BDP}(\bar{M}_B^2, m_\gamma, s, t, u) - f_{\rm IR}(m_\gamma, t, u)$, where

$$f_{\rm IR}(m_\gamma, t, u) = \frac{2}{\pi} B_2(m_\gamma, t) - \frac{2}{\pi} B_2(m_\gamma, u) = \ln\left(\frac{t}{u}\right) \ln\left(\frac{m_\gamma^2}{\sqrt{tu}}\right) + \frac{1}{2} \ln\left(\frac{t}{u}\right).$$

Appendix B: Double bremss. ISR



$$\beta^{(2)}_{2\{11\}}\binom{p\, k_1\, k_2}{\lambda\, \sigma_1\, \sigma_2} = \mathfrak{M}^{(2)}_{2\{11\}}\binom{p\, k_1\, k_2}{\lambda\, \sigma_1\, \sigma_2} - \beta^{(1)}_{1\{1\}}\binom{p\, k_1}{\lambda\, \sigma_1}\mathfrak{s}^{\{1\}}_{[2]} - \beta^{(1)}_{1\{1\}}\binom{p\, k_2}{\lambda\, \sigma_2}\mathfrak{s}^{\{1\}}_{[1]} - \beta^{(0)}_{0}\binom{p}{\lambda}\mathfrak{s}^{\{1\}}_{[1]}\mathfrak{s}^{\{1\}}_{[2]}$$

$$= (eQ_e)^2 \Bigg\{ \frac{\mathfrak{B}_{[ba'][cd]}U_{[a'12'']} - \mathfrak{B}_{[b1'][cd]}U_{[1'12'']} - \mathfrak{B}_{[b2'][cd]}U_{[2'12'']}}{-(1a)-(2a)+(12)} \; \frac{-U_{[2''2a]}}{-(2a)}$$

$$+ \frac{V_{[b11'']}}{-(1b)} \; \frac{-V_{[1''2b']}\mathfrak{B}_{[b'a][cd]} + V_{[1''21']}\mathfrak{B}_{[1'a][cd]} + V_{[1''22']}\mathfrak{B}_{[2'a][cd]}}{-(1b)-(2b)+(12)}$$

$$+ \frac{V_{[b11']}}{-(1b)}\mathfrak{B}_{[1'2'][cd]} \frac{-U_{[2'2a]}}{-(2a)} + (1 \leftrightarrow 2) \Bigg\}$$

$$+ eQ_e \Bigg\{ \frac{-\mathfrak{B}_{[b1'][cd]}U_{[1'1a]} - \mathfrak{B}_{[b2'][cd]}U_{[2'1a]}}{-(1a)-(2a)+(12)} \mathfrak{s}^{(a)}_{[2]} + \mathfrak{s}^{(b)}_{[1]} \frac{V_{[b22']}\mathfrak{B}_{[2'a][cd]} + V_{[b21']}\mathfrak{B}_{[1'a][cd]}}{-(1a)-(2a)+(12)}$$

$$- \mathfrak{s}^{(b)}_{[1]}\mathfrak{B}_{[b2'][cd]}\frac{U_{[2'2a]}}{-(2a)} + \frac{V_{[b11']}}{-(1b)}\mathfrak{B}_{[1'a][cd]}\mathfrak{s}^{(a)}_{[2]} + (1 \leftrightarrow 2) \Bigg\}$$

$$+ \left(\mathfrak{s}^{(a)}_{[1]}\mathfrak{s}^{(a)}_{[2]} \frac{(12)}{(1a)+(2a)-(12)} + \mathfrak{s}^{(b)}_{[1]}\mathfrak{s}^{(b)}_{[2]} \frac{-(12)}{(1b)+(2b)+(12)} \right) \mathfrak{B}\left[{}^{p}_{\lambda}\right].$$

NOTATION : $\mathfrak{s}^{(a)}_{[i]} = -eQ_e \frac{b_{\sigma_1}(k_i, p_a)}{2k_i p_a}$, $\mathfrak{s}^{(b)}_{[i]} = +eQ_e \frac{b_{\sigma_1}(k_i, p_b)}{2k_i p_b}$, $\mathfrak{s}^{\{1\}}_{[i]} \equiv \mathfrak{s}^{(a)}_{[i]} + \mathfrak{s}^{(b)}_{[i]}$

$(ia) \equiv k_i \cdot p_a$, $\quad (ib) \equiv k_i \cdot p_b$. Implicit sums over primed indices!

Conclusions for Part III

- First and second order CEEX is explicitly formulated
- For resonance the coherent treatment of ISR and FSR
- Exact matrix elements up to 2 photons (spinor techniques)
- Full treatment of spin, for beams and decaying final fermions

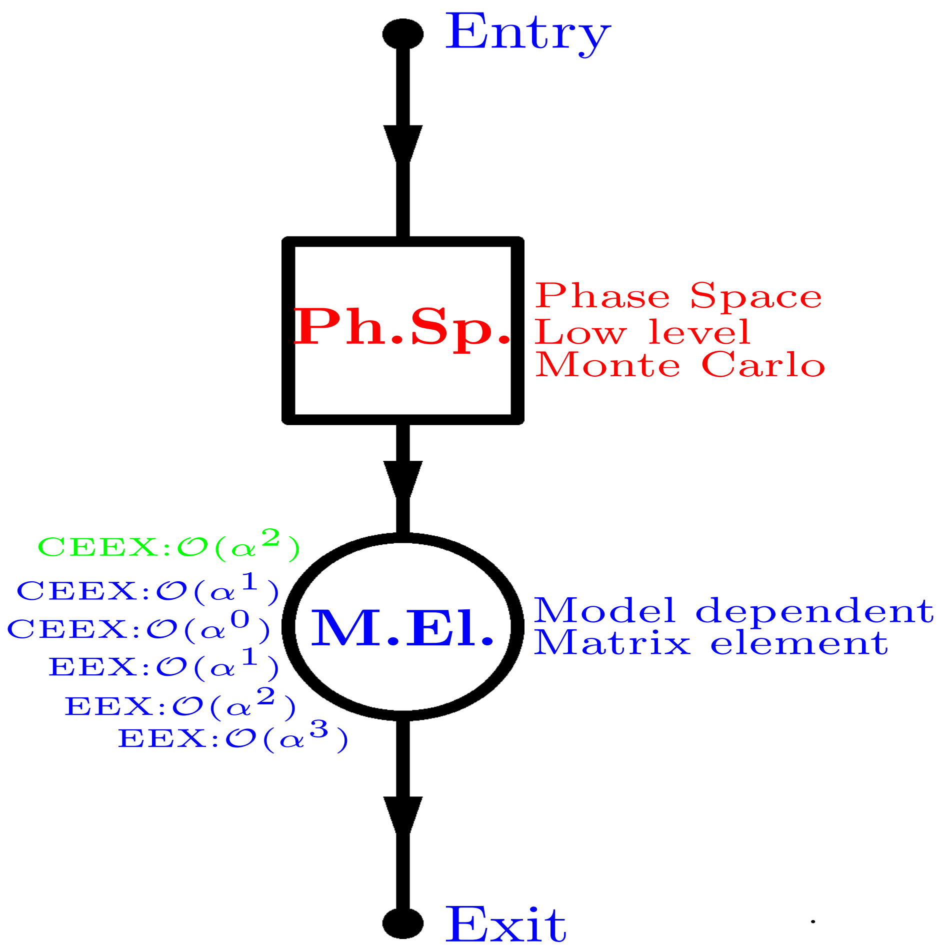
Outline of Part IV

- **General structure of the MC and its precision**
  1. **Basic test of technical precision**
  2. **Basic test of physical precision**
- **ISR$\otimes$FSR interf. in σ_{tot} and A_{FB}**
  1. **How big is ISR$\otimes$FSR interference in σ_{tot}, A_{FB}?**
  2. **Do we know ISR$\otimes$FSR at $\mathcal{O}(\alpha^1)$?**
  3. **Do we know ISR$\otimes$FSR beyond $\mathcal{O}(\alpha^1)$?**
  4. **How sensitive ISR$\otimes$FSR is to cut-off changes?**
  5. **Answers from $\mathcal{KK}$ MC.**
- **Absolute predictions for σ_{tot} A_{FB}**
  1. **Comparisons with KORALZ, $\mathcal{KK}$sem and Zfitter**
  2. **More tests of precision**
- **Summary**

THE PROBLEM: SM predictions for σ_{tot} and A_{FB} for $e^-e^+ \to f\bar{f}$ are needed at the end of LEP2 with the precision $0.2 - 0.5\%$. CAN WE DELIVER?

For example the ISR$\otimes$FSR interference is already $\sim 2\%$.

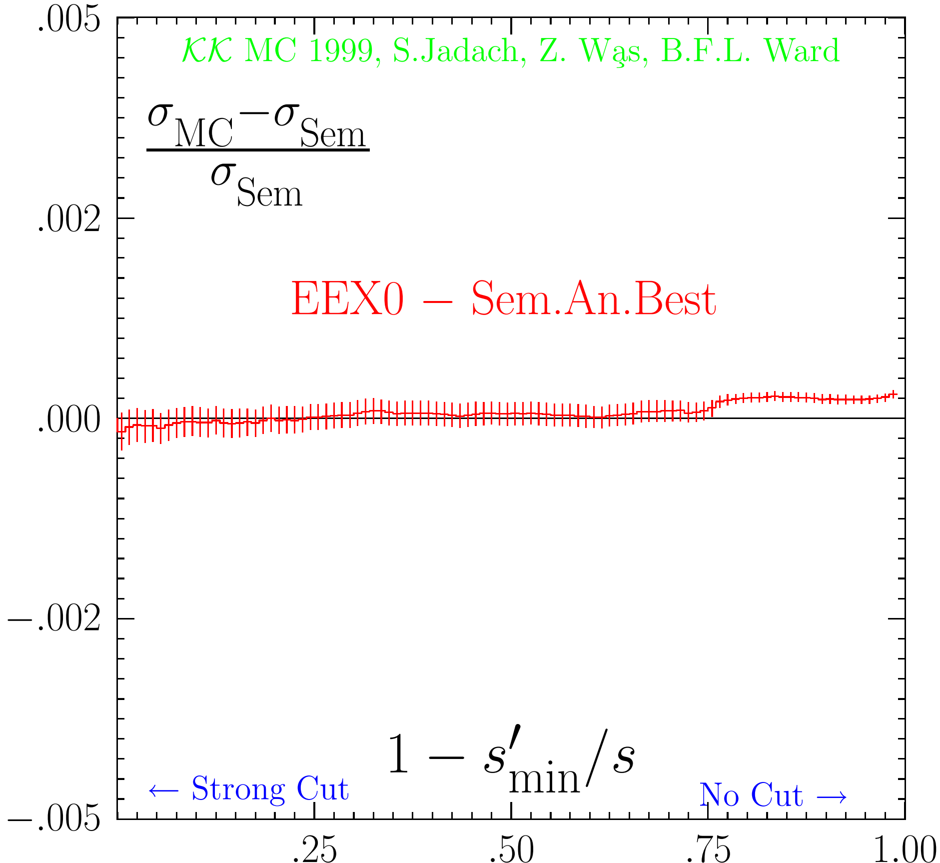
General structure of $\mathcal{KK}$ MC



- **The program for $e^-e^+ \to f\bar{f}$ is divided into two distinct parts/levels**
- **The tau decays and hadronization come after**

Baseline test of the Low level MC

$|$ Matrix Element$|^2$ = Born $\times$ Product of soft factors



$$\sigma_{\rm Sem.} = \int_0^{1-s'_{\min}/s} dv\ \sigma^f_{\rm Born}(s(1-u)(1-v))\ \frac{e^{-C\gamma_e}}{\Gamma(1+\gamma_e)}$$

$$e^{\frac{1}{4}\gamma_e+\frac{\alpha}{\pi}\left(\frac{1}{2}+\frac{\pi^2}{3}\right)}\ \gamma_e v^{\gamma_e-1}\left[1-\frac{\gamma_e\ln(1-v)}{4}-\frac{\alpha}{\pi}\frac{\ln^2(1-v)}{2}+0\ \gamma_e^2\right]$$

$$\frac{e^{-C\gamma_f}}{\Gamma(1+\gamma_f)}e^{\frac{1}{4}\gamma_f-\frac{1}{2}\gamma_f\ln(1-u)+\frac{\alpha}{\pi}\left(\frac{1}{2}+\frac{\pi^2}{3}\right)}\gamma_f u^{\gamma_f-1}\left[1-\frac{\gamma_f\ln(1-u)}{4}\right]$$

Techn.Error=$|$MC−Sem.Anl.$|$ $\leq 2\cdot 10^{-4}$

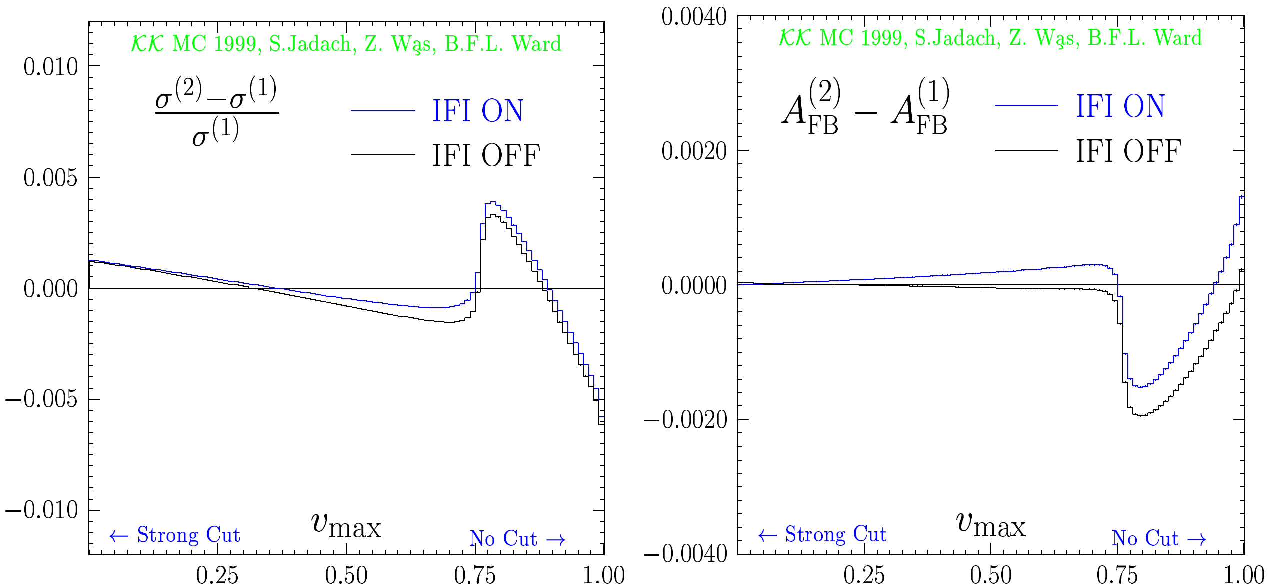
PP = neglected higher orders and subleadings

Our basic estimate: PP = $\frac{1}{2}\left\{\mathcal{O}(\alpha^2)_{\rm CEEX} - \mathcal{O}(\alpha^1)_{\rm CEEX}\right\}$

$\mathcal{KK}$ **MC results for** $e^+e^- \to \mu^+\mu^-$ **at 189GeV**

Photon Energy Cut: $v = 1 - s'/s < v_{\max}$; $s' = m^2_{f\bar{f}}$

Angular Cut: $|\cos\theta| < 1$



The above result imply for Physical Precision:

$\frac{\delta\sigma}{\sigma} \leq 0.2\%$**,** $\delta A_{FB} \leq 0.1\%$**,**

even including radiative-return on-shell Z.

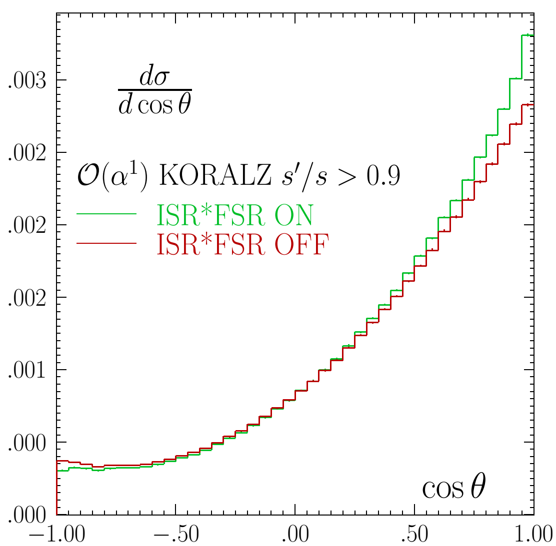
This is confirmed by more tests, see later.

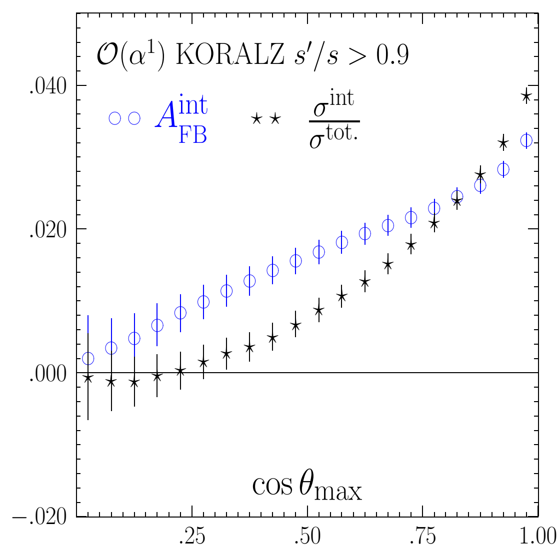
$\mathcal{O}(\alpha^1)$ ISR⊗FSR from KORALZ

KORALZ is the best reference and starting point for the ISR⊗FSR

Angular distributions from KORALZ, pure $\mathcal{O}(\alpha^1)$ (without exponentiation), were verified very precisely $\sim 0.01\%$ using special analytical calculation,

S.Jadach Z.Wąs, Phys. Rev. D41, 1425 (1990).





Results above are for $e^+e^- \rightarrow \mu^+\mu^-$ at $\sqrt{s}$=189GeV.

The energy cut is on s'/s, where $s' = m^2_{f\bar{f}}$.

The angular cut is $|\cos\theta| < \cos\theta_{\rm max}$.

Scattering angle is $\theta = \theta^{\bullet}$,

defi ned in Phys. Rev. **D41**, 1425 (1990)

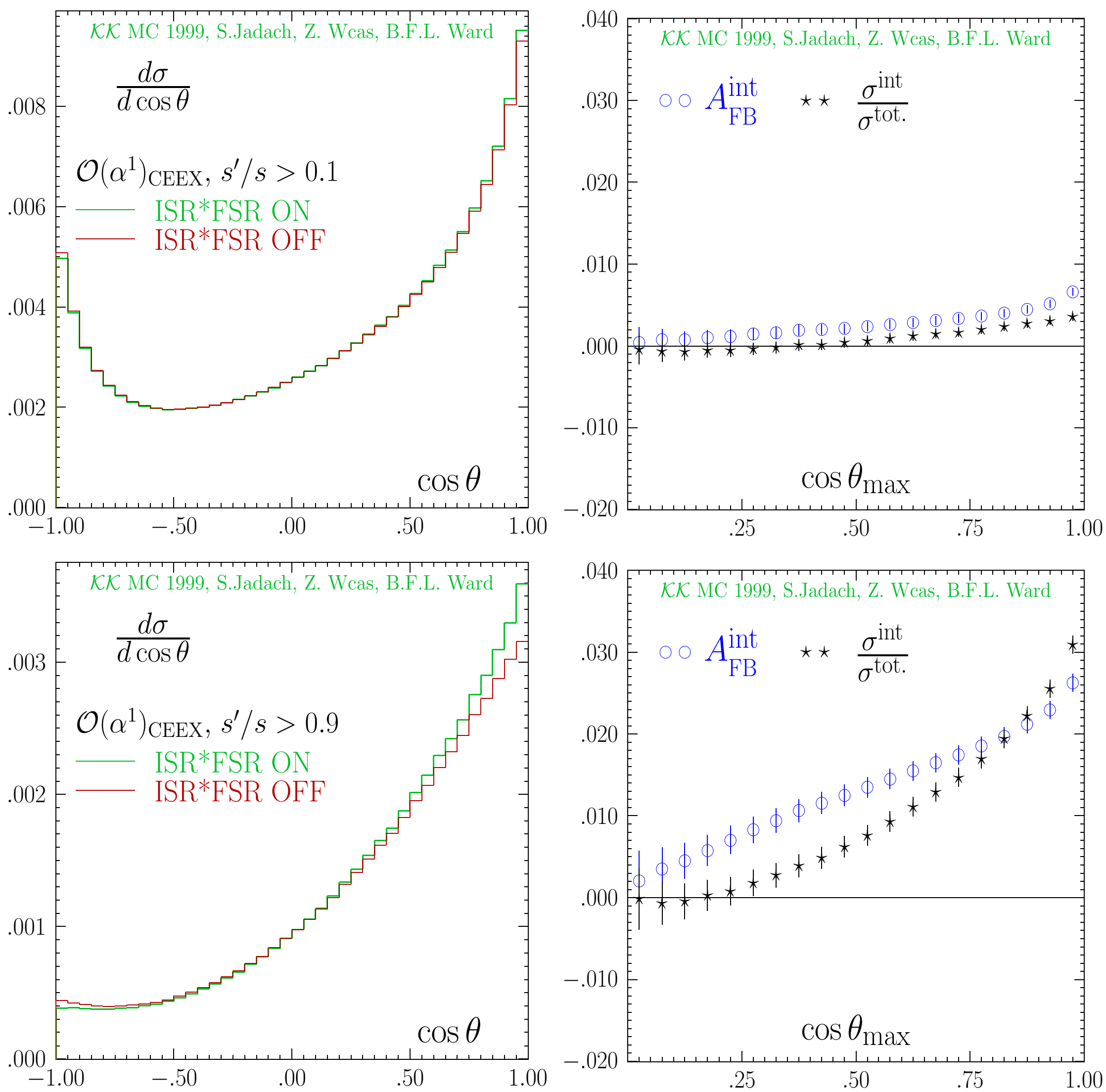
Cut-off Dependence

$\mathcal{KK}$ MC results for $e^+e^- \to \mu^+\mu^-$ at 189GeV

Photon Energy Cut is on $s' = m^2_{f\bar{f}}$

Angular Cut is $|\cos\theta| < \cos\theta_{\max}$

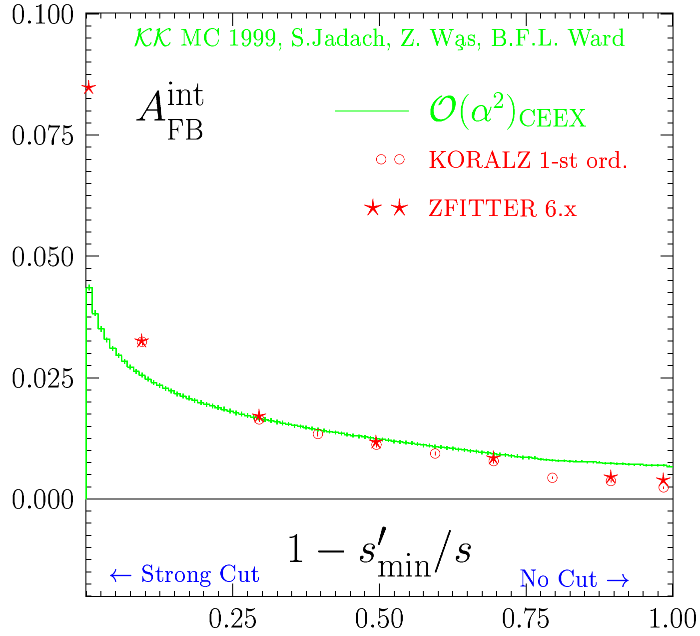
Angle $\theta = \theta^\bullet$ defined in Phys.Rev.**D41**,1425 (1990)



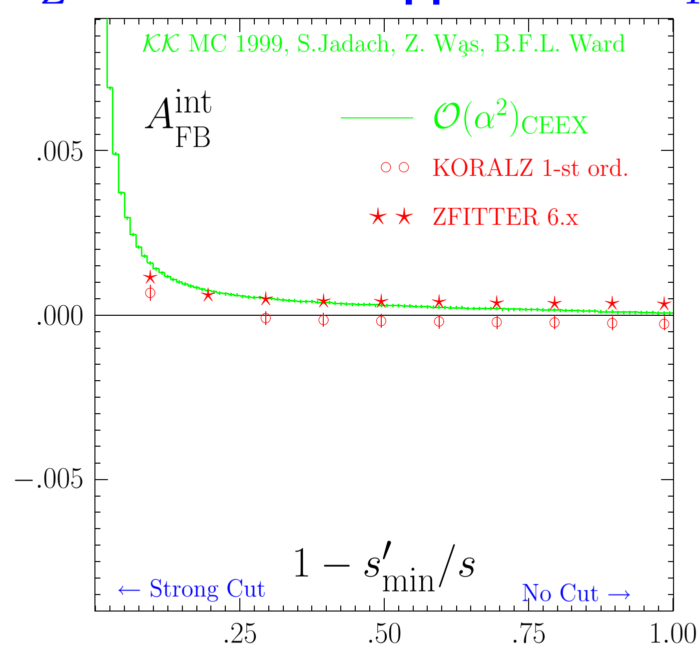
S. Jadach January 3, 2000

s'-cut dependence of δA_{FB}. No θ-cut

At 189GeV the interference corr. δA_{FB} is 2%-5%:



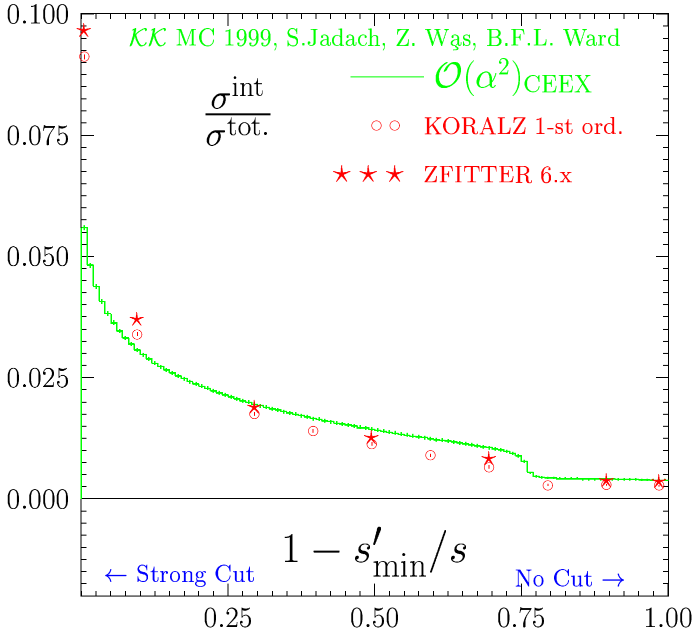
At $\sqrt{s} = M_Z$ the effect is suppressed $\delta A_{FB} < 0.1\%$:



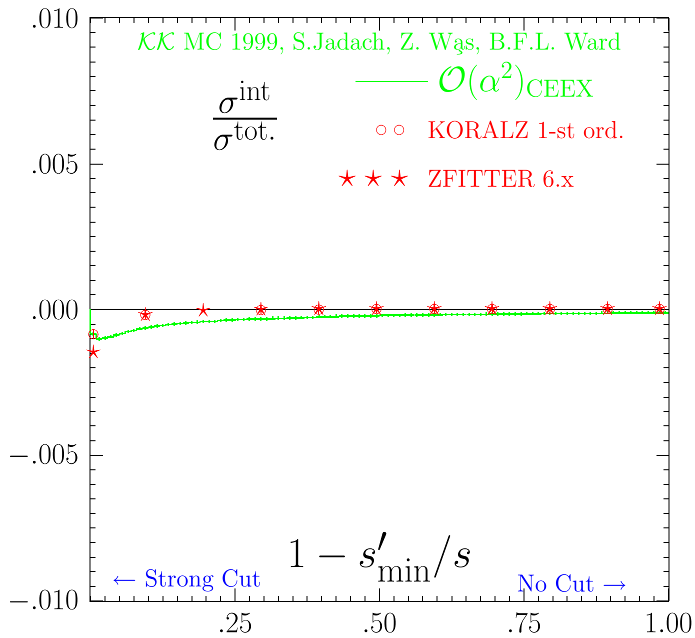
The effect increases strongly with the photon energy cut.

s'-cut dependence of $\delta\sigma$, No θ-cut

At 189GeV interference eff. $\delta\sigma$ is 1%-3%, no $\cos\theta$-cut:



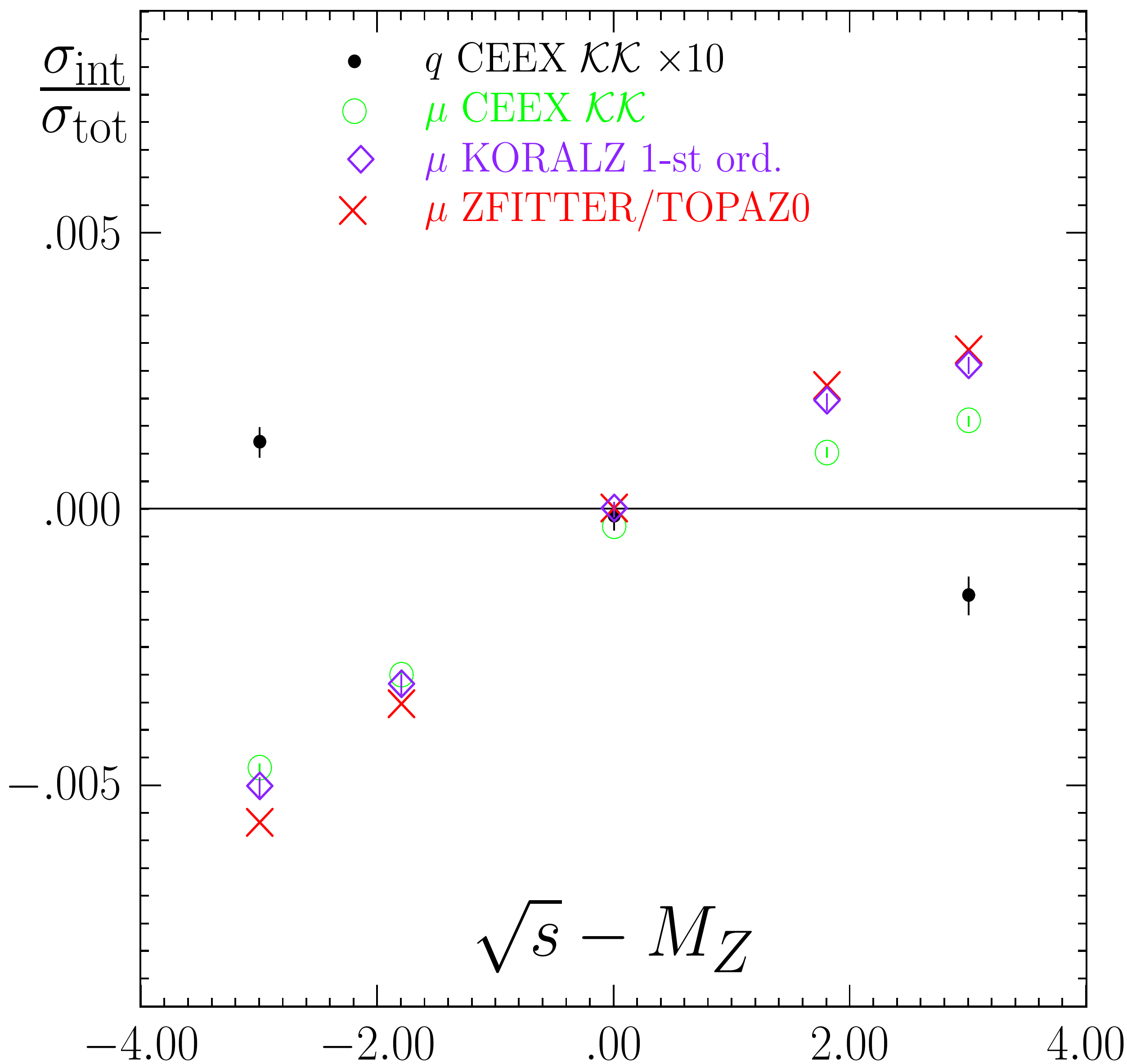
At $\sqrt{s} = M_Z$ the effect is suppressed $\delta\sigma < 0.02\%$:



The effect increases strongly with the photon energy cut.

Back on Z peak

$\mathcal{KK}$ **MC agrees with KORALZ and semi-analytical programs and provides hint about higher orders.**



Initial⊗Final State Interference

$\mathcal{KK}$ Monte Carlo and KORALZ answers on ISR⊗FSR Interference

- **For typical exp. energy cut 0.3 ISR⊗FSR int. is about 1.5% in σ_{tot} and A_{FB}.**
- **For energy cut 0.1 it is twice bigger.**
- **The cut $|\cos\theta < 0.9$ makes it 25% smaller.**
- **The $\mathcal{O}(\alpha^1)$ ISR⊗FSR int. is under total control using KORALZ and $\mathcal{KK}$ Monte Carlo for arbitrary cuts.**
- **Effects beyond $\mathcal{O}(\alpha^1)$ are negligible, (<20% of $\mathcal{O}(\alpha^1)$), except when energy cut is stronger than 0.1.**
- **ISR⊗FSR int. at Z radiative return is very small, as expected.**
- **Change from s' to Q^2-propagator in energy cut has no effect.**

Absolute prediction for σ and A_{FB}

Process: $e^- e^+ \to f\bar{f}$, $f = \mu^-$, **at 189GeV.**

Energy cut: $v < v_{\max}$, **where** $v = 1 - s'/s$, $s' = M^2_{f\bar{f}}$.

Scattering angle for $A_{\rm FB}$ **is** $\theta = \theta^{\bullet}$. **No cut in** θ. **E-W corr. in** $\mathcal{KK}$ **according to DIZET 6.x.** $\mathcal{O}(\alpha^3)_{\rm LL}$ **EEX3 matrix element in** $\mathcal{KK}$ **MC (No ISR⊗FSR interf.)**

$\mathcal{KK}$**sem is semianalytical part of** $\mathcal{KK}$. (Angle $\theta^{\bullet}$ is from Phys. Rev. D41, 1425 (1990).)

$v_{\max}$	$\mathcal{KK}$sem Refer.	$\mathcal{O}(\alpha^3)_{\rm EEX3}$	$\mathcal{O}(\alpha^2)_{\rm CEEX}$ intOFF	$\mathcal{O}(\alpha^2)_{\rm CEEX}$	KORALZ	KORALZ Interf.
	$\sigma(v_{\max})$ [pb], $\mathcal{KK}$ M.C. and KORALZ 1-st order					
0.01	1.6712 ± 0.0000	1.6687 ± 0.0020	1.6669 ± 0.0020	1.7657 ± 0.0024	0.9639 ± 0.0009	0.0912 ± 0.0005
0.10	2.5198 ± 0.0000	2.5164 ± 0.0023	2.5147 ± 0.0023	2.5943 ± 0.0027	2.1919 ± 0.0010	0.0339 ± 0.0004
0.30	3.0616 ± 0.0000	3.0565 ± 0.0024	3.0578 ± 0.0024	3.1183 ± 0.0029	2.7690 ± 0.0010	0.0175 ± 0.0003
0.50	3.3747 ± 0.0000	3.3682 ± 0.0025	3.3740 ± 0.0025	3.4219 ± 0.0029	3.0565 ± 0.0010	0.0113 ± 0.0003
0.70	3.7225 ± 0.0000	3.7131 ± 0.0025	3.7257 ± 0.0025	3.7629 ± 0.0030	3.3649 ± 0.0010	0.0065 ± 0.0003
0.90	7.1434 ± 0.0000	7.0904 ± 0.0024	7.1538 ± 0.0024	7.1795 ± 0.0029	6.3558 ± 0.0010	0.0029 ± 0.0001
0.99	7.6145 ± 0.0000	7.5511 ± 0.0024	7.6642 ± 0.0024	7.6888 ± 0.0029	6.7004 ± 0.0010	0.0028 ± 0.0001
	$A_{\rm FB}(v_{\max})$, $\mathcal{KK}$ M.C. and KORALZ 1-st order					
0.01	0.5654 ± 0.0000	0.5650 ± 0.0014	0.5650 ± 0.0014	0.6111 ± 0.0016	0.5765 ± 0.0013	0.1201 ± 0.0013
0.10	0.5664 ± 0.0000	0.5660 ± 0.0011	0.5660 ± 0.0011	0.5922 ± 0.0012	0.5784 ± 0.0006	0.0324 ± 0.0006
0.30	0.5692 ± 0.0000	0.5687 ± 0.0009	0.5686 ± 0.0009	0.5855 ± 0.0011	0.5818 ± 0.0005	0.0164 ± 0.0005
0.50	0.5744 ± 0.0000	0.5738 ± 0.0009	0.5737 ± 0.0009	0.5862 ± 0.0010	0.5868 ± 0.0005	0.0112 ± 0.0005
0.70	0.5864 ± 0.0000	0.5852 ± 0.0008	0.5853 ± 0.0008	0.5944 ± 0.0009	0.5972 ± 0.0004	0.0078 ± 0.0004
0.90	0.3105 ± 0.0000	0.3115 ± 0.0004	0.3108 ± 0.0004	0.3177 ± 0.0005	0.3260 ± 0.0002	0.0037 ± 0.0002
0.99	0.2851 ± 0.0000	0.2867 ± 0.0004	0.2844 ± 0.0004	0.2903 ± 0.0004	0.3039 ± 0.0002	0.0024 ± 0.0002

Process: $e^- e^+ \to \mu^- \bar{\mu}^-$**, at 189GeV.**

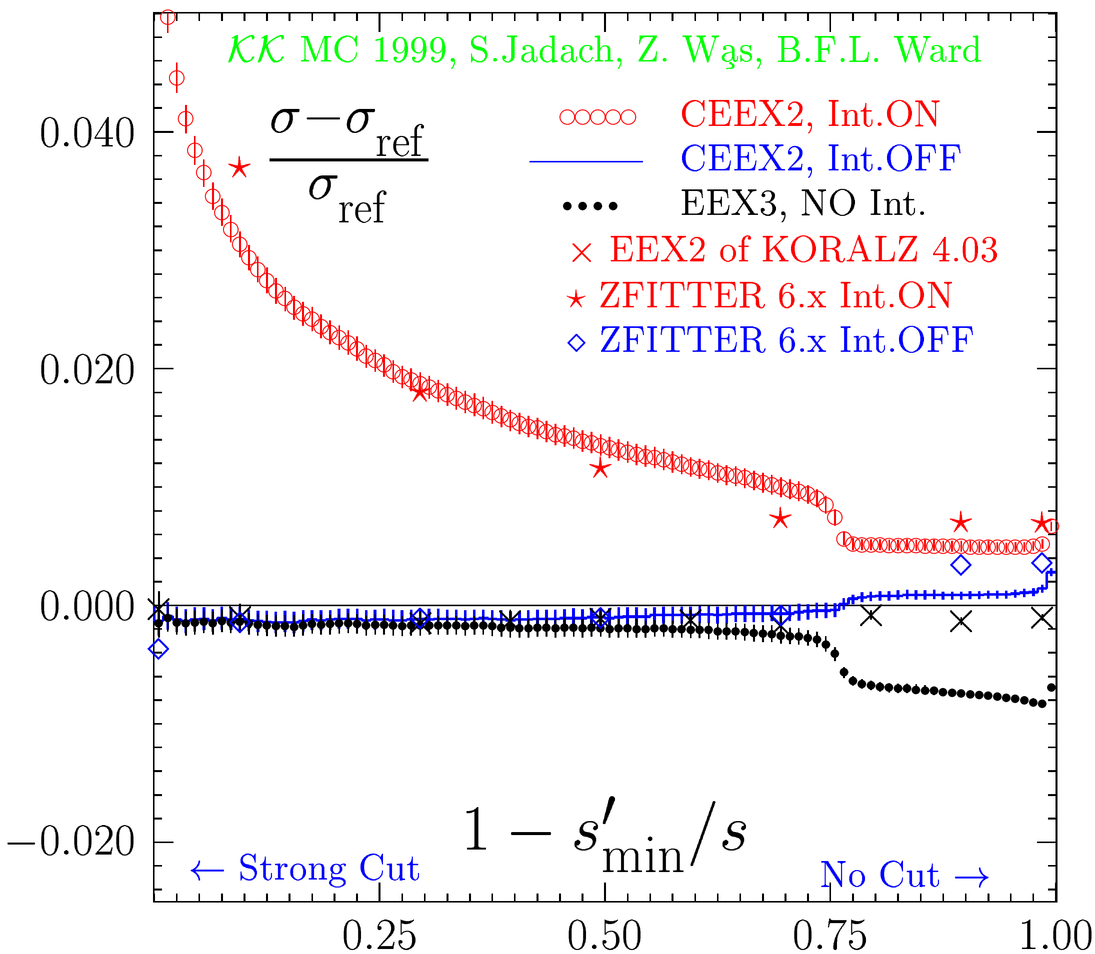
Energy cut: $s' > s'_{\rm min}$**, where** $s' = M^2_{f\bar{f}}$**. No cut in** θ**.**

KORALZ/YFS3 version 4.03, $\mathcal{O}(\alpha^2)_{\rm LL}$**, ISR⊗FSR off.**

ISR⊗FSR interf. on and off, wherever possible.

Reference $\sigma_{\rm ref}$ **is from semi-analytical** $\mathcal{KK}$**SEM,**

ISR⊗FSR off, up to $\mathcal{O}(\alpha^3)_{\rm LL}$**, JSW exponenent.**



S. Jadach January 3, 2000

Process: $e^-e^+ \to \mu^-\bar{\mu}^-$**, at 189GeV.**

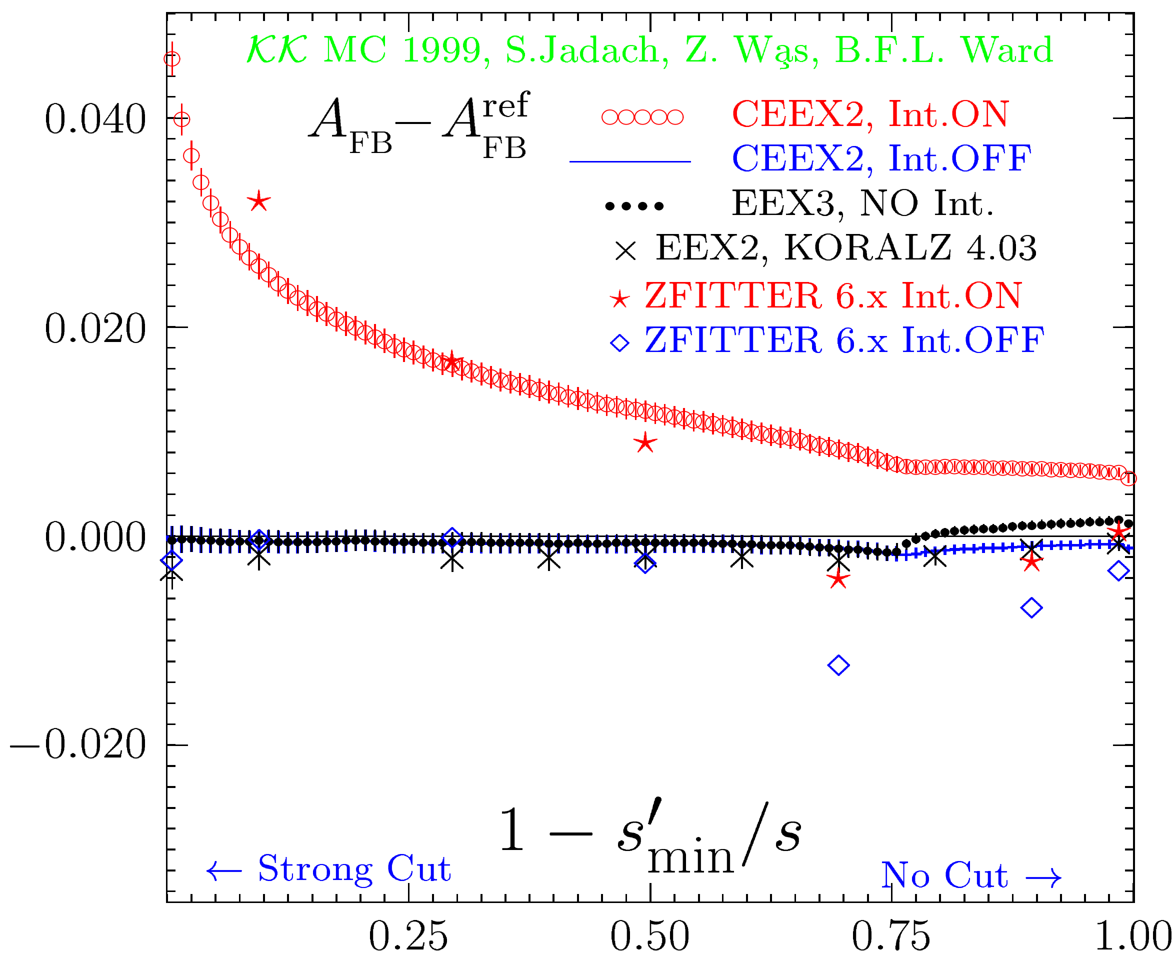
Energy cut: $s' > s'_{\min}$**, where** $s' = M^2_{f\bar{f}}$**. No cut in** θ**.**

KORALZ/YFS3 version 4.03, $\mathcal{O}(\alpha^2)_{\mathrm{LL}}$**, ISR⊗FSR off.**

ISR⊗FSR interf. on and off, wherever possible.

Reference $A^{\mathrm{ref}}_{\mathrm{FB}}$ **is from semi-analytical** $\mathcal{KK}$**SEM,**

ISR⊗FSR off, up to $\mathcal{O}(\alpha^3)_{\mathrm{LL}}$**, JSW exponenent.**



Summary of Part IV

- **Technical Precision** $\frac{\delta\sigma}{\sigma} \leq 2 \cdot 10^{-4}$
- **Physical Precision:**

  $\frac{\delta\sigma}{\sigma} \leq 0.2\%$ **and** $\delta A_{FB} \leq 0.1\%$
- **ISR$\otimes$FSR interf. under control at LEP2 energies**

CEEX Numerical Results

Comparison with our older MC's

Feature	KORALB	KORALZ	$\mathcal{KK}$ now	$\mathcal{KK}$ 2000
QED type	$\mathcal{O}(\alpha)$	EEX	CEEX, EEX	CEEX, EEX
CEEX(ISR+FSR)	none	none	$\{\alpha, \alpha L; \alpha^2 L^2, \alpha^2 L^1\}$	$\{...\alpha^2 L^1; \alpha^3 L^3\}$
EEX(ISR*FSR)	none	$\{\alpha, \alpha L, \alpha^2 L^2\}$	$\{\alpha, \alpha L, \alpha^2 L^2, \alpha^3 L^3\}$	$\{...\alpha^2 L^2, \alpha^3 L^3\}$
ISR-FSR int.	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha)$	$\{\alpha, \alpha L\}_{\rm CEEX}$	$\{\alpha, \alpha L\}_{\rm CEEX}$
Exact bremss.	1 γ	1, 2coll. γ	1, 2, 3coll. γ	up to 3 γ
El-Weak	No Z-res.	DIZET 6.x	DIZET 6.x	YES
Beam polar.	long+trans.	longit.	long+trans.	long+trans.
τ polar.	long+trans.	longit.	long+trans.	long+trans.
Hadronization	—	JETSET	JETSET	PYTHIA
τ decay	TAUOLA	TAUOLA	TAUOLA	TAUOLA
Inclusive mode	—	No	Yes	Yes
Beamstrahlung	—	No	Yes	Yes
beam spread	—	No	Yes	Yes
$\nu\nu$ channel	—	Yes	No	Yes
ee channel	—	No	No	Yes
tt channel	—	No	No	yes?
WW channel	—	No	No	yes?

Conclusions

CEEX offers clear upgrade path for the exclusive exponentiation in the QED, with the Monte Carlo implementation. It is firmly based on spin amplitudes. The main profits are:

- Interferences ISR$\otimes$FSR included and under firm control
- All kind of coherence effects, including narrow resonances.
- Complete treatment of spin (also transverse) for beams and final (unstable) fermions.
- Exact M.E. for 2 high hard photons (3 photon pending).
- Future extension to all-angle second order Bhabha is possible.

First real Monte Carlo implementation is $\mathcal{KK}$ event generator for fermion pair production at LEP, LC's, μ-colliders, τ and b factories.