QED Exponentiation in MC

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The aim is to summarize briefly on:

(1) Main features of IR resumation (YFS) in QED

(2) Systematic inclusion of the order-by-order ME corrections

Papers by S.J., M. Melles, M. Skrzypek, W. Płaczek, B.F.L. Ward, E. Richter-Wąs and Z. Wąs:

Phys. Rev. D **63**, 113009 (2001), Phys. Lett. B449 (1999) 97 and more ...

These and related slides on http://home.cern.ch/jadach

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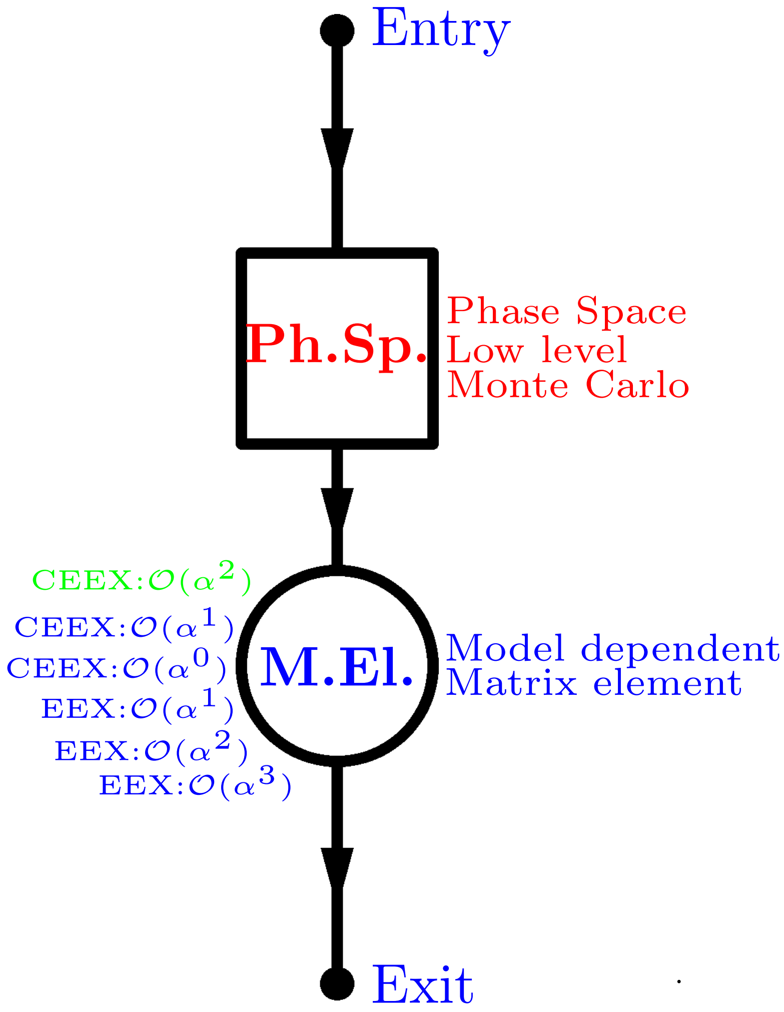
Standard Model calculations for LEP with YFS exponentiation

- $e^+e^- \to f\bar{f} + n\gamma,\ f = \tau, \mu, d, u, s, c,$ YFS1 (1987-1989) $\mathcal{O}(\alpha^1)_{exp}$ ISR,
  YFS2∈KORALZ (1989-1990), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$ ISR
  YFS3∈KORALZ (1990-1998), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$ ISR+FSR
  KKMC (98-02) $\mathcal{O}(\alpha^2 + h.o.LL)_{exp}$ ISR+FSR+Interf. $d\sigma/\sigma = 0.2\%$
- $e^+e^- \to e^+e^- + n\gamma$ for $\theta < 6^\circ$
  BHLUMI 1.x, (1987-1990), $\mathcal{O}(\alpha^1)_{exp}$
  BHLUMI 2.x, (1990-1996), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$ $d\sigma/\sigma = 0.07\%$
- $e^+e^- \to e^+e^- + n\gamma$ for $\theta > 6^\circ$
  BHWIDE (1994-1998), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$
- $e^+e^- \to W^+W^- + n\gamma, W^\pm \to f\bar{f}$
  KORALW (1994-2001)
- $e^+e^- \to W^+W^- + n\gamma, W^\pm \to f\bar{f}$
  YFS3WW (1995-2001), YFS expon. + Leading Pole Approx. $d\sigma/\sigma = 0.4\%$

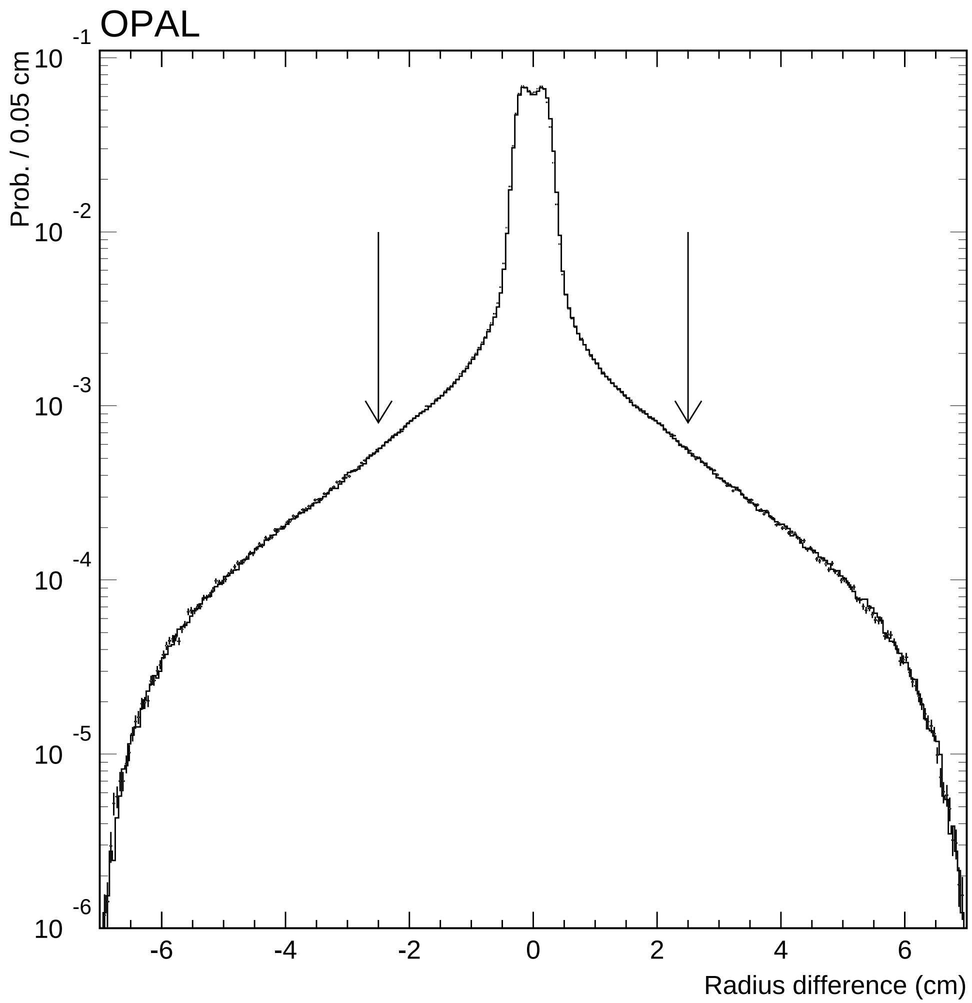
Typical MC realization: "matrix element × exact phase space" principle



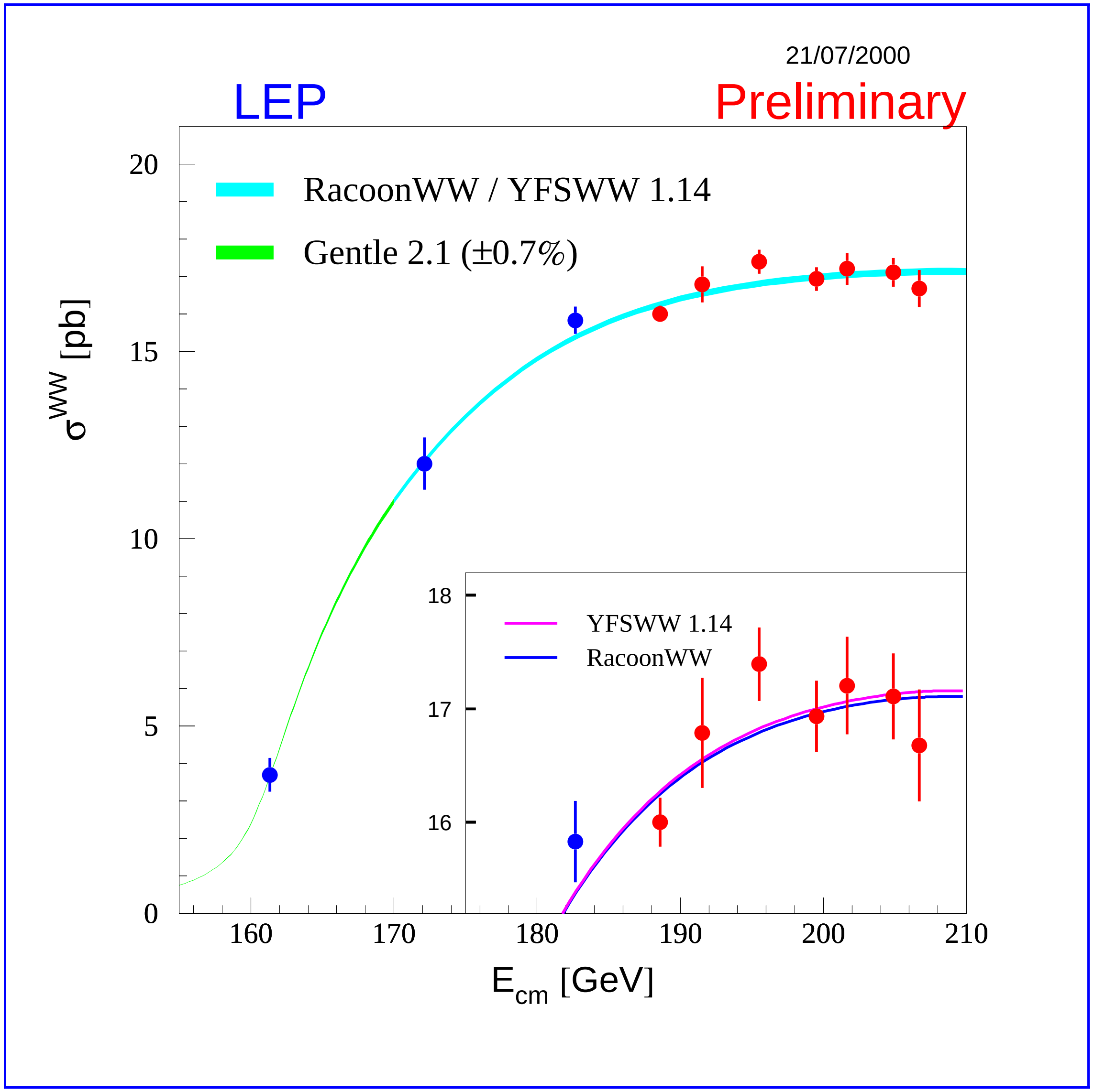
In the Monte Carlo realization it means that:

- Universal exact Phase-space Monte Carlo simulator is a separate module producing "raw events" (with importance sampling)
- Library of several types of SM/QED matrix elements provides "model weight" is another independent module
- Tau decays and hadronization come after of course.

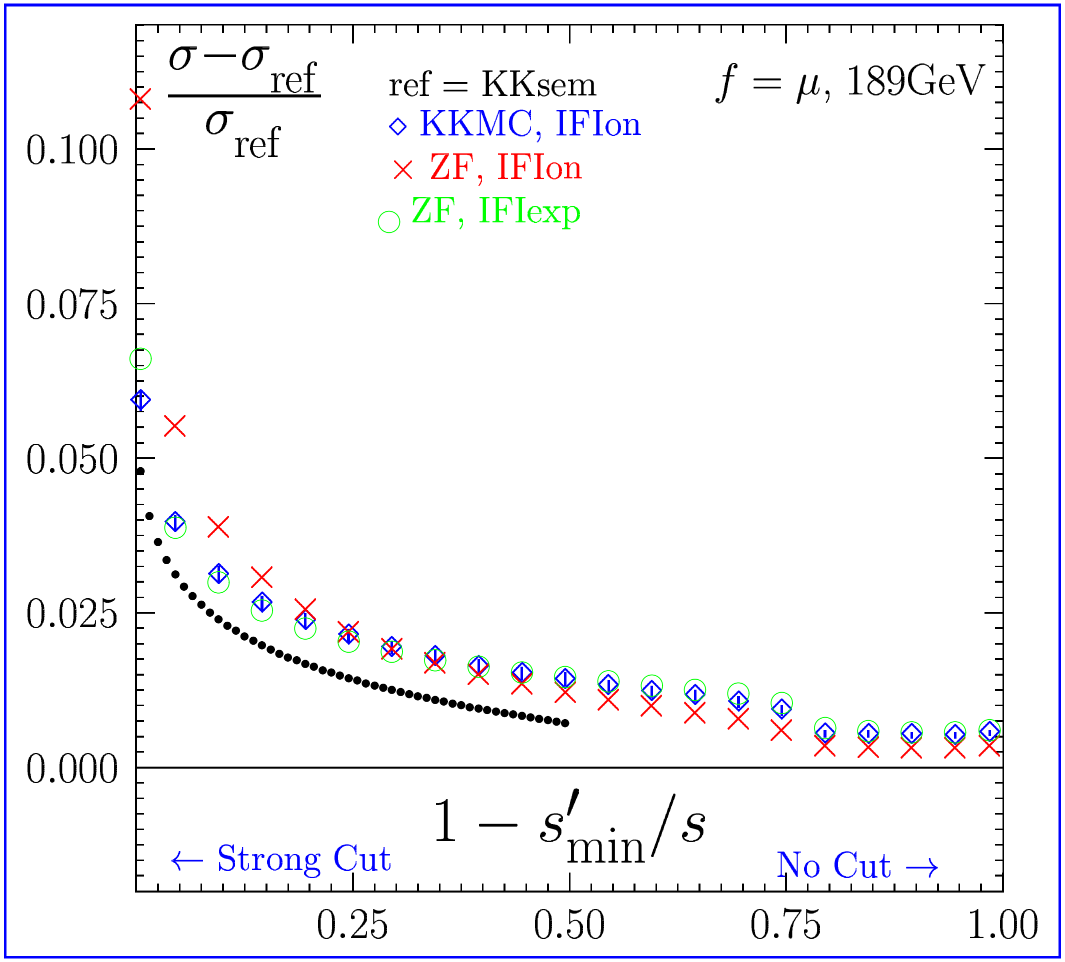
Acollinearity distribution OPAL, MC vs. experiment; $d\sigma/\sigma = 0.07\%$



YFS3WW prediction for WW cross section at LEP2; $d\sigma/\sigma = 0.4\%$



KK MC vs. Zfitter for ISR+FSR, including IFI=ISR⊗FSR; $d\sigma/\sigma = 0.2\%$



IFI of Zfitter legitimized by KKMC! NB. ISR of Zfitter is based on the $\mathcal{O}(\alpha^2)$ calculation by Berends, Burgers, VanNeerven (1984), while KKMC is a totally independent calculation.

Yennie-Frautschi-Suura (1961) main features

YFS technique of IR resummation is based on Lagrangian, Feynman rules, standard renormalization technique, and exact multiphoton phase space integration.

YFS is an example of "factorization theorem", restricted to IR, abelian case.

No structure function. No renormalization group. (RGE helps include h.o. LL corrs.)

"Sudakov double logarithms", i.e. infrared (IR) double logs, $\alpha \ln(s/m^2) \ln(\sqrt{s}/E_{\min})$, summed up to $\mathcal{O}(\alpha^\infty)$

Single non-IR collinear logs $\alpha \ln(s/m^2)$ included order by order.

Starting point is always a finite-order M.E. perturbative calculation, Feynman diagrams.

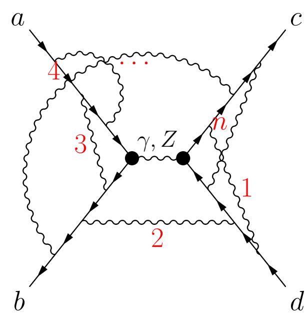
No problem with matching finite-order M.E. and resummed IR parts.

Exact phase space integration essential for inclusion of h.o. exact corrections.

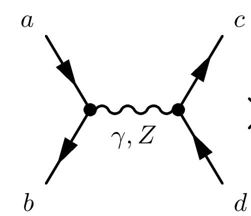
NB. YFS used LL technique for approx. phase space integration – no good computers in 1961! Some readers get wrongly that LL has an essential role in the YFS exponentiation.

Factorization of virtual IR by Yennie-Frautschi-Suura (1961):

$$\sum_{n=0}^{\infty}$$



$$= e^{\alpha B_4}$$



$$\times (1 + \Delta_{\rm finite})$$

where $B_4(p_a, ..., p_d) = \int \frac{d^4 k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} \big| J_I(k) - J_F(k) \big|^2,$

$$J_I = eQ_e\big(\hat{J}_a(k) - \hat{J}_b(k)\big), \quad J_F = eQ_f\big(\hat{J}_c(k) - \hat{J}_d(k)\big), \quad \hat{J}^\mu_f(k) \equiv \frac{2p_f^\mu + k^\mu}{k^2 + 2kp_f + i\epsilon}$$

B_4 **is UV finite because of** k^2 **in the denominator. It is also gauge-invariant.**

The above should be understood, "order by order", as follows:

$\mathcal{O}(\alpha^1)$: $\mathcal{M}^{(1)} = 1 + \alpha B_4 + \alpha \Delta^{(1)}$, where $\Delta^{(1)}$ is IR-finite.

$\mathcal{O}(\alpha^2)$: $\mathcal{M}^{(2)} = 1 + \alpha B_4 + \alpha \Delta^{(1)} + \boxed{\tfrac{1}{2}\alpha^2 B_4^2} + \boxed{\alpha^2 B_4 \Delta^{(1)}} + \alpha^2 \Delta^{(2)}$

$$= \{\exp(\alpha B_4)(1 + \alpha\Delta^{(1)} + \alpha^2 \Delta^{(2)})\}\big|_{\mathcal{O}(\alpha^2)}$$

UNIVERSALITY: At $\mathcal{O}(\alpha^2)$ terms $\boxed{\alpha^2 \ldots}$ are NOT NEW!

Fully determined by $\mathcal{O}(\alpha^1)$. The same at $\mathcal{O}(\alpha^3)$ etc.

YFS Factorisation of real IR singularities proceeds order by order

Consider the case up to $\mathcal{O}(\alpha^2)$ with real photons only:

$\mathcal{O}(\alpha^0)$: $\mathcal{M}^{(0)} = \hat{\beta}_0$,

$\mathcal{O}(\alpha^1)$: $\mathcal{M}^{(1)\mu_1}(k_1) = \hat{\beta}_0 j^{\mu_1}(k_1) + \hat{\beta}_1^{\mu_1}(k_1)$

$\mathcal{O}(\alpha^2)$: $\mathcal{M}^{(2)\mu_1,\mu_2}(k_1,k_2) =$

$$\hat{\beta}_0 j^{\mu_1}(k_1) j^{\mu_2}(k_2) + \hat{\beta}_1^{\mu_1}(k_1) j^{\mu_2}(k_2) + \hat{\beta}_1^{\mu_2}(k_2) j^{\mu_1}(k_1) + \hat{\beta}_2(k_1,k_2)$$

where

$$j_I = eQ_e(\hat{j}_a(k) - \hat{j}_b(k)), \quad j_F = eQ_f(\hat{j}_c(k) - \hat{j}_d(k)), \quad \hat{j}_f^{\mu}(k) \equiv \frac{2p_f^{\mu}}{2kp_f}$$

encapsulate all IR (real) divergences, while $\hat{\beta}_i$ are IR-fi nite.

UNIVERSALITY:

At $\mathcal{O}(\alpha^2)$, $\hat{\beta}_1$ and $\hat{\beta}_0$ and NOT NEW! Determined by $\mathcal{O}(\alpha^1)$ and $\mathcal{O}(\alpha^0)$.

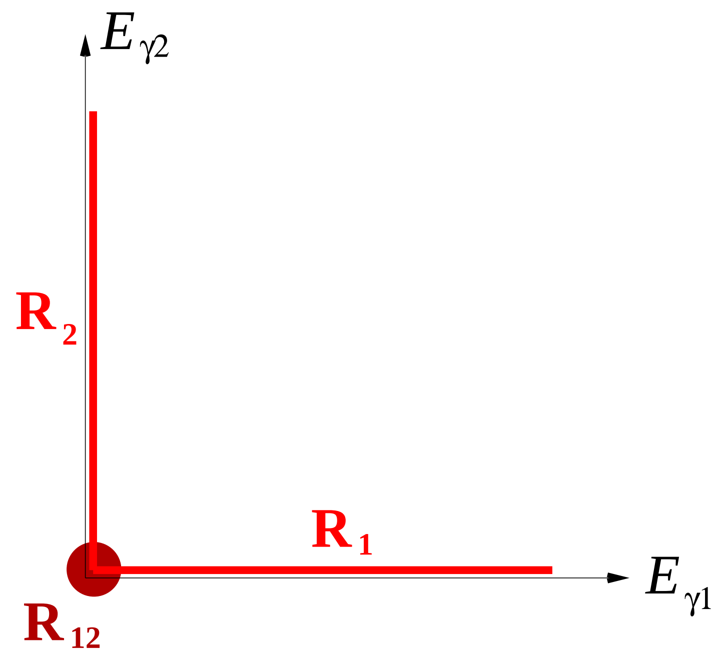
The inductive proof of the above decomposition $\mathcal{O}(\alpha^n) \to \mathcal{O}(\alpha^{n+1})$.

Amplitudes at $\mathcal{O}(\alpha^n)$ for l real and m virtual photons, $n = l + m$, analysed similarly.

Virtual $\exp(\alpha B_4)$ factorises first, decomposition in terms of $j^{\mu}(k)$ done next.

UNIVERSALITY: In $\mathcal{O}(\alpha^{n+1})$ amplitude many components the same as in $\mathcal{O}(\alpha^{n-k})$.

Overlaping divergences: $R_{12} \in R_1$ and $R_{12} \in R_2$



$$D_0(p_{f_1}, p_{f_2}, p_{f_3}, p_{f_4}) = \bar{\beta}_0(p_{f_1}, p_{f_2}, p_{f_3}, p_{f_4}); \qquad p_{f_1} + p_{f_2} = p_{f_3} + p_{f_4}$$
$$D_1(p_f; k_1) = \bar{\beta}_0(p_f)\tilde{S}(k_1) + \bar{\beta}_1(p_f; k_1); \qquad p_{f_1} + p_{f_2} \neq p_{f_3} + p_{f_4}$$
$$D_2(k_1, k_2) = \bar{\beta}_0\tilde{S}(k_1)\tilde{S}(k_2) + \bar{\beta}_1(k_1)\tilde{S}(k_2) + \bar{\beta}_1(k_2)\tilde{S}(k_1) + \bar{\beta}_2(k_1, k_2).$$

IMPORTANT: $\bar{\beta}_0$ and $\bar{\beta}_1$ used beyond their usual (Born and 1γ) phase space.

A kind of smooth "extrapolation" or "projection" is always necessary.

Extra factor (Jacobian or "off-shell") is possible, but *not mandatory*!

Recursive, order-by-order definition of IR-finite $\bar{\beta}$s; SUBTRACTIONS!

$$D_0(p_{f_1},p_{f_2},p_{f_3},p_{f_4}) = \bar{\beta}_0(p_{f_1},p_{f_2},p_{f_3},p_{f_4}),$$
$$D_1(p_f;k_1) = \bar{\beta}_0(p_f)\tilde{S}(k_1) + \bar{\beta}_1(p_f;k_1)$$
$$D_2(k_1,k_2) = \bar{\beta}_0\tilde{S}(k_1)\tilde{S}(k_2) + \bar{\beta}_1(k_1)\tilde{S}(k_2) + \bar{\beta}_1(k_2)\tilde{S}(k_1) + \bar{\beta}_2(k_1,k_2).$$

$$\downarrow$$

$$\bar{\beta}_0(p_{f_1},p_{f_2},p_{f_3},p_{f_4}) = D_0(p_{f_1},p_{f_2},p_{f_3},p_{f_4}),$$
$$\bar{\beta}_1(p_f;k_1) = D_1(p_f;k_1) - \bar{\beta}_0(p_f)\tilde{S}(k_1)$$
$$\bar{\beta}_2(k_1,k_2) = D_2(k_1,k_2) - \bar{\beta}_0\tilde{S}(k_1)\tilde{S}(k_2) - \bar{\beta}_1(k_1)\tilde{S}(k_2) + \bar{\beta}_1(k_2)\tilde{S}(k_1).$$

In the practical order-by-order calculations

IR-finite $\bar{\beta}$s are truncated to certain fixed $\mathcal{O}(\alpha^n)$,

see next slides for $\mathcal{O}(\alpha^1)$ example.

Classsic EEX/YFS schematically; β's truncated to $\mathcal{O}(\alpha^1)$, example of ISR

$$e^-(p_1,\lambda_1) + e^+(p_2,\lambda_2) \to f(q_1,\lambda_1') + \bar{f}(q_2,\lambda_2') + \gamma(k_1,\sigma_1) + \ldots + \gamma(k_n,\sigma_n)$$

$$\sigma = \sum_{n=0}^{\infty} \int_{m_\gamma} d\Phi_{n+2}\, e^{Y(m_\gamma)} D_n(q_1,q_2,k_1,\ldots,k_n)$$

$$D_0 = \bar\beta_0$$
$$D_1(k_1) = \bar\beta_0 \tilde{S}(k_1) + \bar\beta_1(k_1)$$
$$D_2(k_1,k_2) = \bar\beta_0 \tilde{S}(k_1)\tilde{S}(k_2) + \bar\beta_1(k_1)\tilde{S}(k_2) + \bar\beta_1(k_2)\tilde{S}(k_1)$$
$$D_n(k_1,k_2\ldots k_n) = \bar\beta_0 \tilde{S}(k_1)\tilde{S}(k_2)\ldots\tilde{S}(k_n) + \bar\beta_1(k_1)\tilde{S}(k_2)\tilde{S}(k_3)\ldots\tilde{S}(k_n)$$
$$+ \tilde{S}(k_1)\bar\beta_1(k_2)\tilde{S}(k_3)\ldots\tilde{S}(k_n) + \ldots + \tilde{S}(k_1)\tilde{S}(k_2)\tilde{S}(k_2)\ldots\bar\beta_1(k_n)$$

Real soft factors: $\tilde{S}(k) = \sum_\sigma |\mathfrak{s}_\sigma(k)|^2 = |\mathfrak{s}_+|^2(k) + |\mathfrak{s}_-(k)|^2 = -\frac{\alpha}{\pi}\left(\frac{q_1}{kq_1} - \frac{q_2}{kq_2}\right)^2$

IR-finite building blocks:

$\bar\beta_0 = \left(e^{-2\alpha\Re B_4} \sum_\lambda |\mathfrak{M}_\lambda^{\text{Born+Virt.}}|^2\right)\Big|_{\mathcal{O}(\alpha^1)}$, λ = fermion helicities, σ = photon hel.

$$\bar\beta_1(k) = \sum_{\lambda\sigma} |\mathfrak{M}_{\lambda\sigma}^{1-\text{PHOT}}|^2 - \sum_\sigma |\mathfrak{s}_\sigma(k)|^2 \sum_\lambda |\mathfrak{M}_\lambda^{\text{Born}}|^2$$

Everything in terms of $\sum_{spin} |\ldots|^2$! **Distr. < 0 possible for hard 2γ.**

Abandoned MC project, 1989

$$\sigma = \sum_{n=0}^{\infty} \int d\Phi_{n+2}\, D_n(q_1, q_2, k_1, ..., k_n)$$

$$\equiv \sum_{n=0}^{\infty} \int_{m_\gamma} d\Phi_{n+2}\, \bar{\beta}_0(q_1, q_2)\, e^{Y(m_\gamma)} \prod_{i=1}^{n} \tilde{S}(k_i)$$

$$+ \sum_{n=1}^{\infty} \int_{m_\gamma} d\Phi_{n+3}\, \bar{\beta}_1(q_1, q_2, k_1)\, e^{Y(m_\gamma)} \prod_{i=2}^{n} \tilde{S}(k_i)$$

The idea was to generate MC events separately for $\bar{\beta}_0$ and for $\bar{\beta}_1$.

Technically it was feasible but it was abandoned because:

- $\bar{\beta}_1$ can be negative,
- we have found out how to implement method shown in the previous slide.

New CEEX and old EEX, exponentiation schemes derived from YFS 1961 work

EEX= Exclusive EXponentiation, very close to original Yennie-Frautschi-Suura 1961

CEEX = Coherent EXclusive exponentiation, is an extension of YFS

COHERENT = Friendly to Quantum coherence;

Coherence among Feynman diags.

Complete $\left|\sum_{diagr.}^{n} \mathcal{M}_i\right|^2$ rather than often incomplete $\sum_{i,j}^{n^2} \mathcal{M}_i \mathcal{M}_j^*$.

Narrow resonances, $\gamma \oplus Z$ exchanges, $t \oplus s$ channels, ISR$\oplus$FSR, Angular ordering, etc.

CEEX schematically, ISR $\mathcal{O}(\alpha^1)$ Example:

$$e^-(p_1,\lambda_1)+e^+(p_2,\lambda_2)\to f(q_1,\lambda_1')+\bar{f}(q_2,\lambda_2')+\gamma(k_1,\sigma_1)+\ldots+\gamma(k_n,\sigma_n)$$

$$\sigma=\sum_{n=0}^{\infty}\int_{m_\gamma} d\Phi_{n+2}\sum_{\lambda,\sigma_1,\ldots,\sigma_n}|e^{\alpha B(m_\gamma)}\mathcal{M}^{\lambda}_{n,\sigma_1,\ldots,\sigma_n}(k_1,\ldots,k_n)|^2$$

$\mathcal{M}^{\lambda}_0=\hat{\beta}^{\lambda}_0$, λ=fermion helicities,

$$\mathcal{M}^{\lambda}_{1,\sigma_1}(k_1)=\hat{\beta}^{\lambda}_0\mathfrak{s}_{\sigma_1}(k_1)+\hat{\beta}^{\lambda}_{1,\sigma_1}(k_1)$$

$$\mathcal{M}^{\lambda}_{2,\sigma_1,\sigma_2}(k_1,k_2)=\hat{\beta}^{\lambda}_0\mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)+\hat{\beta}^{\lambda}_{1,\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)+\hat{\beta}^{\lambda}_{1,\sigma_2}(k_2)\mathfrak{s}_{\sigma_1}(k_1)$$

$$\mathcal{M}^{\lambda}_{n,\sigma_1,\ldots\sigma_n}(k_1,k_2,\ldots k_n)=\hat{\beta}^{\lambda}_0\mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n)+\hat{\beta}^{\lambda}_{1,\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n)$$
$$+\mathfrak{s}_{\sigma_1}(k_1)\hat{\beta}^{\lambda}_{1,\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n)+\ldots+\mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\hat{\beta}^{\lambda}_{1,\sigma_2}(k_2)$$

IR-finite building blocks:

$$\hat{\beta}^{\lambda}_0=\left(e^{-\alpha B_4}\mathcal{M}^{\text{Born+Virt.}}_{\lambda}\right)\Big|_{\mathcal{O}(\alpha^1)},$$

$$\hat{\beta}^{\lambda}_{1,\sigma}(k)=\mathcal{M}^{\lambda}_{1,\sigma}(k)-\hat{\beta}^{\lambda}_0\mathfrak{s}_{\sigma}(k)$$

Everything in terms of $\mathcal{M}$-spin-amplitudes!

Distr. ≥ 0 by construction!

Another useful feature of YFS/CEEX

Flexibility in the choice of the IR regulator in the CEEX/YFS

The IR cancellations do occur independently in two places:

[a] between the exponential formfactor and the real-photon $\int PhaseSpace$

[b] between the various term *inside* the well defined IR-finite β**-functions.**

Hence, a freedom to choose different IR regulators (a) and (b).

Case (a) of IR cancellations: YFS formfactor vs. real γ integrals

$$\sigma = \sum_{n=0}^{\infty} \int_{m_\gamma} d\Phi_{n+2} \sum_{\lambda,\sigma_1,\dots,\sigma_n} |e^{\alpha B(m_\gamma)} \mathfrak{M}^{\lambda}_{n,\sigma_1,\dots,\sigma_n}(k_1,\dots,k_n)|^2$$

Here, for $\int d^3k$ going into $D \neq 4$ dimensions makes little sense.

One may choose fi nite photon mass or any other convenient "photon energy cut" method which works at $D = 4$.

Case (b) of IR cancellations: in construction and evaluation of βs

2 kinds of cancellations: real-real and virtual-virtual (no real-virt.)

- The real-real ones occur for the integrand of the real-emission phase space before $\int d^3k$ integration, even better, they occur for the spin amplitudes, before taking square!
  Example: $\hat{\beta}^{\lambda}_{1,\sigma}(k) = \mathcal{M}^{\lambda}_{1,\sigma}(k) - \hat{\beta}^{\lambda}_{0}\mathfrak{s}_{\sigma}(k)$.
  No need for any IR regulation, it works entirely numerically.
- In virtual components of the β's, the numerical virtual-virtual IR cancellations can be executed before $\int d^D k$ integration.
  $\hat{\beta}^{\lambda}_{0} = \left(e^{-\alpha B}\mathcal{M}^{\rm Born+Virt.}_{\lambda}\right)\Big|_{\mathcal{O}(\alpha^1)} = \mathcal{M}^{\rm Born+Virt.}_{\lambda} - \alpha B\ \mathcal{M}^{\rm Born}_{\lambda}$.
  The IR-regulator may be even unnecessary!
  For D-regularization the cancellation of term $1/\epsilon$ of IR origin always done after $\int d^D k$ integration.

Conclusions

- **YFS inspired EEX and CEEX schemes are succesful examples of the Monte Carlos based directly on the factorization theorem, albeit for IR soft case for abelian QED only.**
- **Work well in practice: KORALZ, BHLUMI, YWSWW3, BHWIDE, KK MC.**
- **May simplify the next generation of the precision exponentiated $\mathcal{O}(\alpha^2)$ calculations (Bhabha!) and the corresponding MCs for Linear Colliders.**
- **Extension (as far as possible) to all collinear singularities would be very desirable and practicaly important!**